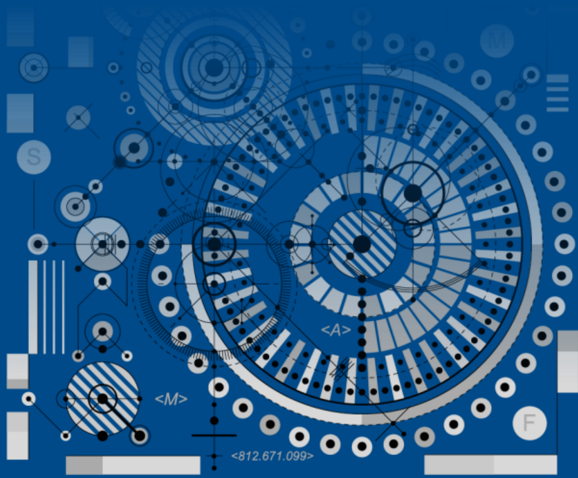
Research memoir on belief reliability



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Preface

Reliability is defined as the capability that a product can perform a required function for a given period of time under specified operating conditions. How to measure reliability is a basic problem in reliability science.

For a long time, researchers and engineers have used probability theory to measure reliability. The method of obtaining this metric, which we call probabilistic reliability, is based on the statistical inference of product failure time data. However, this metric has a lot of problems in the actual application scenarios, wherefore many researchers have tried to correct the probabilistic reliability metric. Till now, people have successively proposed reliability metrics based on Bayesian theory, interval analysis, evidence theory, fuzzy sets and possibility theory, but there are still various problems with these metrics.

We believe that a normative and scientific reliability metric must satisfy four requirements, i.e., slow decrease, self-duality, multiscale analysis, and uncertain information fusion. Based on these four requirements, we propose a new metric called *belief reliability* based on probability theory, uncertainty theory and chance theory.
This proceedings contains 15 papers published during the establishing process of belief reliability theory in the past 5 years. It shows theoretical and application research papers including reviews, theoretical framework, information fusion, design optimization etc.

We sincerely welcome academic and engineering peers to criticize, discuss and use the belief reliability theory.

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Contents

Belief reliability for uncertain random systems

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***Abstract*—Measuring system reliability by a reasonable metric is a basic problem in reliability engineering. Since the real systems are usually uncertain random systems which is affected by both aleatory and epistemic uncertainties, the existed reliability metrics may not work well. This paper aims to develop a general reliability metric, called belief reliability metric, to cope with the problem. In this paper, the belief reliability is defined as the chance that a system state is within a feasible domain. Mathematically, the metric can degenerate to either probability theory-based reliability, which mainly copes with aleatory uncertainty, or uncertainty theory-based reliability, which mainly considers the effect of epistemic uncertainty. Based on the proposed metric, some commonly used belief reliability indexes, such as belief reliability distribution, mean time to failure and belief reliable life, are introduced. We also develop some system belief reliability formulas for different systems configurations. To further illustrate the formulas, a real case study is finally performed in this paper.**

***Index Terms*—Belief reliability, Chance theory, Uncertain random system, Reliability metric.**

I. INTRODUCTION

RELIABILITY is one of the most important properties of systems. It refers to the capability that a component or system can perform a required function for a given period of time under stated operating conditions [1]. In reliability engineering, quantifying reliability by a quantitative metric is a fundamental problem. Only on the basis of a reasonable reliability metric can we better carry out reliability design, reliability analysis and reliability assessments. The key problem for determining a reliability metric is how to cope with uncertainties affecting products. In general, there are two types of uncertainties: aleatory uncertainty caused by inherent randomness of the physical world, and epistemic uncertainty coming from our lack of knowledge about a system [2], [3].

Traditional reliability metrics are based on probability theory. At the very beginning, the reliability of a product is calculated using statistical methods to analyze the product's failure time data [4], [5]. Based on the law of large numbers, the acquisition of this reliability metric requires large samples of failure time data. Since our knowledge about the system is included in the data, reflected as the system-to-system variations of failure times, there is no need to strictly distinguish the two types of uncertainties. However, in the product development process, it is often difficult to collect enough statistical data of the system failure time. This promots the physical model-based reliability metric, where the failure of a system is regarded to be determined by physics-of-failure (PoF) models [6], [7]. Through PoF models, the reliability of a system can be improved by eliminating weak points. In this method, the parameters in the PoF models are usually described as probability distributions, which reflects the effect of aleatory uncertainty. The system reliability can be, then, obtained by propagating the aleatory uncertainty through the PoF models [8]. However, because of our limited information of the products, the process of selecting or establishing a PoF model is often influenced by epistemic uncertainty, and this is especially true in the design of innovative products. Without accounting for the effect of epistemic uncertainty, this metric may overestimate the reliability of a system [9]. Therefore, to better design and improve system reliability, people tend to consider the two types of uncertainties separately.

Considering the effect of epistemic uncertainty, many reliability metrics are proposed based on different mathematical theories, such as evidence theory-based reliability metric [10], [11], interval analysis-based reliability metric [12], [13], fuzzy interval analysis-based reliability metric [14] and posbist reliability metric [15]. Among the four reliability metrics, the first three are all given as reliability intervals, and the last one is defined as a possibility measure. As pointed out by Kang et al. [16], the reliability interval-based metrics may cause interval extension problems when calculating system reliability and posbist reliability does not satisfy duality property which may lead to counter-intuitive results.

For this reason, a new mathematical theory called uncertainty theory is utilized to measure system reliability. Uncertainty theory was founded by Liu [17] in 2007 and refined by Liu [18] in 2010. By introducing the uncertain measure, the uncertain variable, the uncertainty distribution and other concepts, uncertainty theory is viewed as an appropriate mathematical system to model epistemic uncertainty [19], [16]. To simulate the evolution of uncertain phenomena over time, researchers also proposed the tools of uncertain process [20] and uncertain differential equation [21]. After these years of development, uncertainty theory has been applied in various areas, including uncertain finance [22], [23], decision making [24], [25], uncertain control [26], maintenance optimization [27], etc.

In 2010, Liu [28] first described system reliability as an uncertain measure mathematically and proposed a reliability index theorem to calculate the reliability of boolean systems. Later in 2013, Zeng et al. [29] name this reliability metric as *belief reliability*, and interpreted the metric as the belief degree of the system to be reliable. They also clarified the strong need for a new metric in reliability engineering, and proposed a significant belief reliability analysis method for applications. Since the theoretical basis of this reliability metric is uncertainty theory [30], in this paper, we call this metric uncertainty theory-based reliability for convenience.

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Because of the axioms of uncertainty theory, this metric satisfy duality property and will not cause interval extension problems. The uncertainty theory-based reliability metric seems to be more proper to measure system reliability considering epistemic uncertainty. Nowadays, however, real engineering systems are usually consist of different types of components. Some components may suffer great epistemic uncertainty and their reliabilities are modeled by uncertainty theory, while others may be mainly affected by aleatory uncertainty and their reliabilities are measured based on probability theory. We call these kind of real systems, in this paper, as uncertain random systems. Obviously, the reliability of an uncertain random system cannot be analyzed based only on probability theory or only on uncertainty theory.

To address this problem, a chance theory proposed by Liu [31] in 2013 is introduced in this paper. Chance theory can be regarded as a mixture of probability theory and uncertainty theory, and can be utilized to describe systems with both randomness and uncertainty. In chance theory, the uncertain random variable and the chance distribution are two fundamental concepts. To describe uncertain random events over time, uncertain random process was proposed [32], [33]. In recent years, chance theory has developed steadily and applied widely in many fields such as uncertain random risk analysis [34], portfolio optimization [35] and project scheduling [36].

The reliability metric based on chance theory is first introduced by Wen and Kang [37] to measure the reliability of an uncertain random Boolean system. However, the metric has the following shortcomings. First, it does not consider the effect of time, which is a significant factor in reliability engineering. Second, only the reliability of systems with two states is defined, and the multi-state system reliability is not given. Thirdly, they only discuss the system reliability mathematically, and some physical meanings are still insufficient.

To avoid the above shortcomings, in this paper, we aim to expand the connotation of *belief reliability* and develop a general definition of belief reliability metric. The general metric, of course, can degenerate to both traditional probability theory-based reliability metric and uncertainty theory-based belief reliability metric. In addition, for the need of engineering applications, the new reliability metric can be estimated by means of either failure time data, performance margin or system function level. Some system belief reliability formulas are also discussed in this paper.

The remainder of this paper are structured as follows. Section II introduces some mathematical basis of uncertainty theory and chance theory. In Section III, the belief reliability metric is defined and discussed based on chance theory. Some important belief reliability indexes in reliability engineering are defined in Section IV. In Section V, formulas for system belief reliability are given for simple and complex systems, respectively. A real case study about the reliability analysis of an apogee engine is performed to illustrate the formulas. Finally, some conclusions are made in Section VI.

II. Preliminary

In this section, some basic concepts and results of uncertainty theory and chance theory are introduced.

A. Uncertainty theory

Uncertainty theory is a new branch of axiomatic mathematics built on four axioms, i.e., Normality, Duality, Subadditivity and Product Axioms. Founded by Liu [17] in 2007 and refined by Liu [18] in 2010, uncertainty theory has been widely applied as a new tool for modeling subjective (especially human) uncertainties. In uncertainty theory, belief degrees of events are quantified by defining uncertain measures:

Definition II.1 (Uncertain measure [17])**.** Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms,

Axiom 1 (Normality Axiom)**.** *$\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*

Axiom 2 (Duality Axiom)**.** *$\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$.*

Axiom 3 (Subadditivity Axiom)**.** *For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\left\{\Lambda_i\right\}.$$

Uncertain measures of product events are calculated following the product axiom [38]:

Axiom 4 (Product Axiom)**.** *Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\left\{\Lambda_k\right\},$$

where $\mathcal{L}_k$ are σ-algebras over Γ_k, and Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition II.2 (Uncertain variable [17])**.** An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in \mathcal{B}\}$ is an event for any Borel set $\mathcal{B}$ of real numbers.

Definition II.3 (Independence [38])**.** The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Definition II.4 (Uncertainty distribution [17])**.** The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x.

An uncertainty distribution Φ is said to be regular if it is a continuous and strictly increasing with respect to x, with $0 < \Phi(x) < 1$, and $\lim_{x\to-\infty}\Phi(x) = 0$, $\lim_{x\to+\infty}\Phi(x) = 1$. A regular uncertainty distribution has an inverse function, which is defined as the inverse uncertainty distribution, denoted by $\Phi^{-1}(\alpha)$, $\alpha \in (0, 1)$. Inverse uncertainty distributions play a central role in uncertainty theory, since the uncertainty distribution of a function of uncertain variables is calculated using the inverse uncertainty distributions:

Theorem II.1 (Operational law [18]). *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution*

$$\begin{aligned}\Psi^{-1}(\alpha) =& f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \\ & \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).\end{aligned} \tag{II.1}$$

Definition II.5 (Expected value [17]). Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}dx. \tag{II.2}$$

The expected value of an uncertain variable can be also calculated using its uncertainty distribution or inverse uncertainty distribution.

Theorem II.2. [17] *Let ξ be an uncertain variable with an uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx.$$

Theorem II.3. [18] *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha.$$

B. Chance theory

Chance theory is founded by Liu [31], [39] as a mixture of uncertainty theory and probability theory, to deal with problems affected by both aleatory uncertainty (randomness) and epistemic uncertainty. The basic concept in chance theory is the chance measure of an event in a chance space.

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a chance space.

Definition II.6 (chance measure [31]). Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Then the chance measure of Θ is defined as

$$\text{Ch}\{\Theta\} = \int_0^1 \Pr\left\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\right\} dx. \tag{II.3}$$

Theorem II.4. [31] *Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space. Then*

$$\text{Ch}\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times \Pr\{A\} \tag{II.4}$$

for any $\Lambda \in \mathcal{L}$ and any $A \in \mathcal{A}$. Especially, we have

$$\text{Ch}\{\varnothing\} = 0, \quad \text{Ch}\{\Gamma \times \Omega\} = 1. \tag{II.5}$$

Definition II.7 (Uncertain random variable [31]). An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B of real numbers.

Random variables and uncertain variables are two special cases of uncertain random variables. If an uncertain random variable $\xi(\gamma, \omega)$ does not vary with γ, it degenerates to a random variable. If an uncertain random variable $\xi(\gamma, \omega)$ does not vary with ω, it degenerates to an uncertain variable.

Example II.1. Let $\eta_1, \eta_2, \cdots, \eta_m$ be random variables and $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. If f is a measurable function, then

$$\xi = f\left(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n\right)$$

is an uncertain random variable determined by

$$\begin{aligned}\xi(\gamma, \omega) =& f(\eta_1(\omega), \eta_2(\omega), \cdots, \eta_m(\omega), \\ & \tau_1(\gamma), \tau_2(\gamma), \cdots, \tau_n(\gamma))\end{aligned}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Definition II.8. Let ξ be an uncertain random variable. Then its chance distribution is defined by

$$\Phi(x) = \text{Ch}\{\xi \leq x\} \tag{II.6}$$

for any $x \in \Re$.

Example II.2. As a special uncertain random variable, the chance distribution of a random variable η is just its probability distribution, that is,

$$\Phi(x) = \text{Ch}\{\eta \leq x\} = \Pr\{\eta \leq x\}.$$

Example II.3. As a special uncertain random variable, the chance distribution of an uncertain variable τ is just its uncertainty distribution, that is,

$$\Phi(x) = \text{Ch}\{\tau \leq x\} = \mathcal{M}\{\tau \leq x\}.$$

Theorem II.5. [39] *Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. Assume f is a measurable function. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has a chance distribution

$$\begin{aligned}\Phi(x) = \int_{\Re^m} & F(x; y_1, y_2, \cdots, y_m) \\ & d\Psi_1(y_1)d\Psi_2(y_2)\cdots d\Psi_m(y_m),\end{aligned} \tag{II.7}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$.

Definition II.9. [31] Let ξ be an uncertain random variable. Then its expected value is defined by

$$E[\xi] = \int_0^{+\infty} \text{Ch}\{\xi \geq x\}dx - \int_{-\infty}^{0} \text{Ch}\{\xi \leq x\}dx, \tag{II.8}$$

provided that at least one of the two integrals is finite.

Theorem II.6. [31] *Let ξ be an uncertain random variable with chance distribution Φ. Then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx. \tag{II.9}$$

If $\Phi(x)$ is a regular chance distribution, we can calculate the expected value by means of the inverse distribution [31]:

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \tag{II.10}$$

Definition II.10. Let ξ be an uncertain random variable with finite expected value e. Then the variance of ξ is

$$V[\xi] = E[(\xi - e)^2]. \tag{II.11}$$

Since $(\xi - e)^2$ is a nonnegative uncertain random variable, we also have

$$V[\xi] = \int_0^{+\infty} \text{Ch}\{(\xi - e)^2 \geq x\}dx. \tag{II.12}$$

III. Belief reliability metric

In this section, we will introduce the belief reliability metric to measure reliability of uncertain random systems. Usually, we are interested in the state of a system under given conditions. Therefore, we define a state variable which is able to describe the system function or failure behaviors. In an uncertain random system, the states of some components are modeled as random variables and those of other components are described as uncertain variables. Therefore, the state variable of the system will be an uncertain random variable. When the system state variable is within a feasible domain, which reflects our tolerance degree of failure, the system is regarded to be reliable under this state. Based on this, the belief reliability is defined as follows.

Definition III.1 (Belief reliability)**.** Let a system state variable ξ be an uncertain random variable, and Ξ be the feasible domain of the system state. Then the belief reliability is defined as the chance that the system state is within the feasible domain, i.e.,

$$R_B = \text{Ch}\{\xi \in \Xi\}. \tag{III.1}$$

Remark III.1. If the state variable ξ degenerate to a random variable, the belief reliability metric will be a probability. Let $R_B^{(P)}$ denotes the belief reliability under probability theory. Then

$$R_B = R_B^{(P)} = \Pr\{\xi \in \Xi\}. \tag{III.2}$$

This means the system is mainly influenced by aleatory uncertainty, and the belief reliability degenerate to the probability theory-based reliability metric.

Remark III.2. If the state variable ξ degenerate to an uncertain variable, the belief reliability metric will be a belief degree. Let $R_B^{(U)}$ denotes the belief reliability under uncertainty theory. Then

$$R_B = R_B^{(U)} = \mathcal{M}\{\xi \in \Xi\}. \tag{III.3}$$

This means the system is mainly influenced by epistemic uncertainty, and the belief reliability degenerate to the uncertainty theory-based reliability metric.

In Definition III.1, the state variable ξ describes the system behavior, while the feasible domain Ξ is a reflection of failure criteria. In reliability engineering, ξ is a physical quantity that can be measured or predicted through tests, physical models or online monitoring. Ξ is usually described mathematically as a subset of real numbers (eg., an interval) which include the acceptable values of ξ. When the value of ξ falls in Ξ, we say the system is working well, otherwise we say the system fails. For example, ξ may represent the system performance margin and $\xi > 0$ means the system is working, then correspondingly, Ξ will be interval $(0, +\infty)$. To better demonstrate belief reliability metric and the physical meaning of ξ and Ξ, we will offer 3 examples in the following parts. Another point that should be emphasized is that both ξ and Ξ can be relevant to time t, since the system behavior and the failure criteria usually vary with time in practice. Therefore, the belief reliability is usually a function of t, denoted as $R_B(t)$, which is regarded as belief reliability function in this paper.

Example III.1. The state variable can be the system failure time T which describes system failure behaviors. The system is regarded to be reliable at time t if the failure time is greater than t. Thus, the belief reliability of the system at time t can be obtained by letting the feasible domain of T to be $(t, +\infty)$. In this case, the belief reliability will be written as the form with respect to the failure time:

$$R_B(t) = \text{Ch}\{T > t\}. \tag{III.4}$$

If the system is mainly affected by aleatory uncertainty, the system failure time will be modeled as a random variable $T^{(P)}$. The system belief reliability becomes

$$R_B(t) = R_B^{(P)}(t) = \Pr\{T^{(P)} > t\}.$$

Similarly, if the system contains great epistemic uncertainty, the system failure time will be described as an uncertain variable $T^{(U)}$. The system belief reliability becomes

$$R_B(t) = R_B^{(U)}(t) = \mathcal{M}\{T^{(U)} > t\}.$$

Example III.2. The state variable can represent the performance margin m (defined by Zeng et al. [40]) of a system, which describes system function behaviors. m indicates the distance between a performance parameter and the associated failure threshold. A failure will occur if $m < 0$ and $m = 0$ indicates an unstable critical state. Therefore, the system feasible domain, in this case, should be $(0, +\infty)$ and the system belief reliability can be written as the form with respect to the performance margin:

$$R_B = \text{Ch}\{m > 0\}. \tag{III.5}$$

If we consider the degradation process of the performance margin, i.e., the state variable is relevant to t, $R_B(t)$ will be

$$R_B(t) = \text{Ch}\{m(t) > 0\}. \tag{III.6}$$

The failure time T and the performance margin m are the most commonly used system state variable in reliability engineering. Considering the effect of time, the function behavior represented by m will finally convert to the failure behavior described by T. Therefore, the meanings expressed by the two forms of belief reliability metric (III.4) and (III.6) are consistent. Actually, $m(t)$ is an uncertain random process with a threshold level of 0. The first hitting time of $m(t)$ will be

$$t_0 = \inf\{t \geq 0 | m(t) = 0\} \tag{III.7}$$

with a chance distribution of $\Upsilon(t)$. Since t_0 is just the failure time of the product, we have

$$R_B(t) = \text{Ch}\{m(t) > 0\} = 1 - \Upsilon(t) = \text{Ch}\{T > t\}. \tag{III.8}$$

Example III.3. If we consider a multi-state system, the state variable should be the system function level, denoted as G, which describes the behavioral status of a system as it performs its specified function. Assume the system has k different function levels $G = i, i = 0, 1, \cdots, k$ with a lowest acceptable level of $G = s$. Let $G = k$ represent the system functions perfectly, and $G = i, i = s, s+1, \cdots, k-1$ reflect the different degraded working states, then the system belief reliability can be obtained by letting the system feasible domain to be $\{s, s+1, \cdots, k\}$, i.e.,

$$R_B = \text{Ch}\left\{G \in \{s, s+1, \cdots, k\}\right\}. \tag{III.9}$$

The effect of time, of course, can be also considered by assuming the system function level vary with time, that is

$$R_B(t) = \text{Ch}\left\{G(t) \in \{s, s+1, \cdots, k\}\right\}. \tag{III.10}$$

For Eq. (III.9), if the system only has two function levels, namely, complete failure with $G = 0$ and perfectly function with $G = 1$, the system belief reliability will just be the metric proposed by Wen and Kang [37], i.e.,

$$R_B = \text{Ch}\left\{G = 1\right\}. \tag{III.11}$$

Therefore, the reliability metric they developed is exactly a special case of the belief reliability defined in this paper.

IV. Some belief reliability indexes

In this section, some commonly used belief reliability indexes, including belief reliability distribution, belief reliable life, mean time to failure and variance of failure time, are defined based on the belief reliability metric.

Definition IV.1 (Belief Reliability Distribution)**.** Assume that a system state variable ξ is an uncertain random variable, then the chance distribution of ξ, i.e.,

$$\Phi(x) = Ch\{\xi \leq x\}, \tag{IV.1}$$

is defined as the belief reliability distribution.

Example IV.1. If ξ represents the product failure time T, the belief reliability distribution will be the chance distribution of T, denoted as $\Phi(t) = \text{Ch}\{T \leq t\}$. In this case, the sum of $\Phi(t)$ and $R_B(t)$ equals 1, i.e.,

$$\Phi(t) + R_B(t) = 1. \tag{IV.2}$$

Example IV.2. If ξ represents the system performance margin m, the belief reliability distribution will be the chance distribution of m, denoted as $\Phi(x) = \text{Ch}\{m \leq x\}$.

Definition IV.2 (Belief Reliable Life)**.** Assume the system failure time T is an uncertain random variable with a belief reliability function $R_B(t)$. Let α be a real number from $(0, 1)$. The system belief reliable life $T(\alpha)$ is defined as

$$T(\alpha) = \sup\{t | R_B(t) \geq \alpha\}. \tag{IV.3}$$

Example IV.3. BL1 life is defined as

$$t_{BL1} = T(0.99) = \sup\{t | R_B(t) \geq 0.99\},$$

which is one of the most commonly used belief reliable life. It means that the systems have a chance of 0.99 to survive till this time.

Example IV.4. The median time to failure is also commonly used in reliability engineering, which is defined as

$$t_{\text{med}} = T(0.5) = \sup\{t | R_B(t) \geq 0.5\}.$$

Apparently, to identify whether a system has the chance to work till t_{med} is the most difficult.

Definition IV.3 (Mean Time to Failure, MTTF)**.** Assume the system failure time T is an uncertain random variable with a belief reliability function $R_B(t)$. The mean time to failure (MTTF) is defined as

$$\text{MTTF} = E[T] = \int_0^\infty \text{Ch}\{T > t\} dt = \int_0^\infty R_B(t) dt. \tag{IV.4}$$

Theorem IV.1. *Let $R_B(t)$ be a continuous and strictly decreasing function with respect to t at which $0 < R_B(t) < R_B(0) \leq 1$ and $\lim_{t \to +\infty} R_B(t) = 0$. If $T(\alpha)$ is defined by (IV.3), then we have*

$$\textit{MTTF} = \int_0^1 T(\alpha) d\alpha. \tag{IV.5}$$

Proof. Assume the belief reliability distribution of T is $\Phi(t)$, then we have $R_B(t) = \text{Ch}\{T > t\} = 1 - \text{Ch}\{T < t\} = 1 - \Phi(t)$. Since $R_B(t)$ has inverse function, $\Phi(t)$ has an inverse distribution $\Phi^{-1}(\alpha)$. It follows from (IV.3) that

$$T(\alpha) = \sup\{t | \Phi(t) \leq 1 - \alpha\} = \Phi^{-1}(1 - \alpha). \tag{IV.6}$$

Thus, MTTF can be written as

$$\begin{aligned} \text{MTTF} = E[T] &= \int_0^1 \Phi^{-1}(\alpha) d\alpha \\ &= \int_0^1 \Phi^{-1}(1 - \alpha) d\alpha = \int_0^1 T(\alpha) d\alpha. \end{aligned} \tag{IV.7}$$

Definition IV.4 (Variance of failure time, VFT)**.** Assume the system failure time T is an uncertain random variable and the mean time to failure is MTTF. The variance of failure time (VFT) is defined as

$$\text{VFT} = V[T] = E\left[(T - \text{MTTF})^2\right]. \tag{IV.8}$$

Theorem IV.2. *Let the belief reliability function be $R_B(t)$, then the VFT can be calculated by*

$$\begin{aligned} \textit{VFT} = \int_0^{+\infty} &(R_B(MTTF + \sqrt{t}) + \\ &1 - R_B(MTTF - \sqrt{t})) dt. \end{aligned} \tag{IV.9}$$

Proof. Since $(T-\text{MTTF})^2$ is a nonnegative uncertain random variable, we have

$$\begin{aligned}\text{VFT} &= \int_0^{+\infty} \text{Ch}\{(T-MTTF)^2 \geq t\}dt \\ &= \int_0^{+\infty} \text{Ch}\{(T \geq MTTF+\sqrt{t}) \cup \\ &\qquad (T \leq MTTF-\sqrt{t})\}dt \\ &\leq \int_0^{+\infty} (\text{Ch}\{T \geq MTTF+\sqrt{t}\}+ \\ &\qquad \text{Ch}\{T \leq MTTF-\sqrt{t}\})dt \\ &= \int_0^{+\infty} (R_B(MTTF+\sqrt{t})+ \\ &\qquad 1-R_B(MTTF-\sqrt{t}))dt.\end{aligned} \tag{IV.10}$$

In this case, we stipulate that VFT takes the maximum value in (IV.10), i.e.,

$$\begin{aligned}\text{VFT} = \int_0^{+\infty} &(R_B(MTTF+\sqrt{t})+ \\ &1-R_B(MTTF-\sqrt{t}))dt.\end{aligned} \tag{IV.11}$$

V. System belief reliability fomulas

This section will propose some system belief reliability formulas. In this paper, we only discuss the situation that the system state variable ξ is failure time. The circumstances that ξ stands for performance margin or function level can be studied similarly.

Here, in an uncertain random system, the components mainly affected by aleatory uncertainty are regarded as random components whose failure times are described as random variables, while those mainly influenced by epistemic uncertainty are called uncertain components with failure times represented by uncertain variables.

A. Belief reliability formula for simple systems

Sometimes, an uncertain random system can be simplified to be composed of two types of subsystems — a random subsystem only including random components and an uncertain subsystem only containing uncertain components, and the two types of subsystems will be connected in either series or parallel. We do not strictly require the configurations inside the two subsystems and they can be very complex. The belief reliability of this kind of systems can be also calculated based on the following theorems.

Theorem V.1. *Assume an uncertain random system is simplified to be composed of a random subsystem with belief reliability $R_{B,R}^{(P)}(t)$ and an uncertain subsystem with belief reliability $R_{B,U}^{(U)}(t)$. If the two subsystems are connected in series, the system belief reliability $R_{B,S}(t)$ will be*

$$R_{B,S}(t) = R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t). \tag{V.1}$$

Proof. Assume the failure times of the random components and uncertain components in the two types of subsystems are $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively. According to the configurations of the subsystems, the failure times of the random subsystem $T_R^{(P)}$ and the uncertain subsystem $T_U^{(U)}$ are determined by $T_R^{(P)} = f(\eta_1, \eta_2, \cdots, \eta_m)$ and $T_U^{(U)} = g(\tau_1, \tau_2, \cdots, \tau_n)$, respectively, where f and g are two measurable functions. Therefore, $T_R^{(P)}$ is a random variable and $T_U^{(U)}$ is an uncertain variable. Since the two subsystems are connected in series, the system failure time can be written as:

$$T = T_R^{(P)} \wedge T_U^{(U)}.$$

Then we have

$$\begin{aligned}R_{B,S}(t) &= \text{Ch}\{T > t\} \\ &= \text{Ch}\{T_R^{(P)} \wedge T_U^{(U)} > t\} \\ &= \text{Ch}\left\{\left(T_R^{(P)} > t\right) \cap \left(T_U^{(U)} > t\right)\right\} \\ &= \text{Pr}\left\{T_R^{(P)} > t\right\} \times \mathcal{M}\left\{T_U^{(U)} > t\right\} \\ &= R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t).\end{aligned} \tag{V.2}$$

Example V.1 (Series system)**.** Consider an uncertain random series system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Then the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with series configurations, and the two subsystems are connected in series. Assume the failure times of random and uncertain components are $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively. Then the belief reliability of the system $R_{B,S}(t)$ can be calculated according to Theorem V.1:

$$\begin{aligned}R_{B,S}(t) &= R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t) \\ &= \text{Pr}\left\{\bigwedge_{i=1}^{m} \eta_i > t\right\} \times \mathcal{M}\left\{\bigwedge_{j=1}^{n} \tau_j > t\right\} \\ &= \text{Pr}\left\{\bigcap_{i=1}^{m} (\eta_i > t)\right\} \times \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j > t)\right\} \\ &= \prod_{i=1}^{m} R_{B,i}^{(P)}(t) \cdot \bigwedge_{j=1}^{n} R_{B,j}^{(U)}(t).\end{aligned} \tag{V.3}$$

Example V.2 (Parallel series system)**.** Consider an uncertain random parallel series system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Suppose the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with parallel configurations, and the two subsystems are connected in series. By assuming the failure times and random and uncertain components to be $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively, the belief reliability of the system $R_{B,S}(t)$ can be calculated according to Theorem V.1:

$$\begin{aligned} R_{B,S}(t) &= R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t) \\ &= \Pr\left\{\bigvee_{i=1}^{m} \eta_i > t\right\} \times \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j > t\right\} \\ &= \left(1 - \Pr\left\{\bigvee_{i=1}^{m} \eta_i \le t\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j \le t\right\}\right) \\ &= \left(1 - \Pr\left\{\bigcap_{i=1}^{m} (\eta_i \le t)\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j \le t)\right\}\right) \\ &= \left(1 - \prod_{i=1}^{m} (1 - R_{B,i}^{(P)}(t))\right) \cdot \bigvee_{j=1}^{n} R_{B,j}^{(U)}(t). \end{aligned} \tag{V.4}$$

Theorem V.2. *Assume an uncertain random system is simplified to be composed of a random subsystem with belief reliability $R_{B,R}^{(P)}$ and an uncertain subsystem with belief reliability $R_{B,U}^{(U)}$. If the two subsystems are connected in parallel, the system belief reliability will be*

$$R_{B,S}(t) = 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right). \tag{V.5}$$

Proof. Similar to the proof of Theorem V.1, the failure times of random subsystem $T_R^{(P)}$ and uncertain subsystem $T_U^{(U)}$ are random variable and uncertain variable, respectively. Since the two subsystems are connected in parallel, the system failure time can be written as:

$$T = T_R^{(P)} \vee T_U^{(U)}.$$

Then we have

$$\begin{aligned} R_{B,S}(t) &= \text{Ch}\{T > t\} \\ &= \text{Ch}\{T_R^{(P)} \vee T_U^{(U)} > t\} \\ &= 1 - \text{Ch}\left\{T_R^{(P)} \vee T_U^{(U)} \le t\right\} \\ &= 1 - \text{Ch}\left\{\left(T_R^{(P)} \le t\right) \cap \left(T_U^{(U)} \le t\right)\right\} \\ &= 1 - \Pr\left\{T_R^{(P)} \le t\right\} \times \mathcal{M}\left\{T_U^{(U)} \le t\right\} \\ &= 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right). \end{aligned} \tag{V.6}$$

Example V.3 (Parallel system)**.** Consider an uncertain random parallel system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Then the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with parallel configurations, and the two subsystems are connected in parallel. Assume the failure times of random and uncertain components are $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively. Then the belief reliability of the system $R_{B,S}(t)$ can be calculated according to Theorem V.2:

$$\begin{aligned} R_{B,S}(t) &= 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right) \\ &= 1 - \left(1 - \Pr\left\{\bigvee_{i=1}^{m} \eta_i > t\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j > t\right\}\right) \\ &= 1 - \Pr\left\{\bigvee_{i=1}^{m} \eta_i \le t\right\} \times \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j \le t\right\} \\ &= 1 - \Pr\left\{\bigcap_{i=1}^{m} (\eta_i \le t)\right\} \times \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j \le t)\right\} \\ &= 1 - \left(\prod_{i=1}^{m} (1 - R_{B,i}^{(P)}(t))\right) \cdot \left(1 - \bigvee_{j=1}^{n} R_{B,j}^{(U)}(t)\right). \end{aligned} \tag{V.7}$$

Example V.4 (Series parallel system)**.** Consider an uncertain random series parallel system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Suppose the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with series configurations, and the two subsystems are connected in parallel. By assuming the failure times and random and uncertain components to be $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively, the belief reliability of the system $R_{B,S}(t)$ can be calculated according to Theorem V.2:

$$\begin{aligned} R_{B,S}(t) &= 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right) \\ &= 1 - \left(1 - \Pr\left\{\bigwedge_{i=1}^{m} \eta_i > t\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigwedge_{j=1}^{n} \tau_j > t\right\}\right) \\ &= 1 - \left(1 - \Pr\left\{\bigcap_{i=1}^{m} (\eta_i > t)\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j > t)\right\}\right) \\ &= 1 - \left(1 - \prod_{i=1}^{m} R_{B,i}^{(P)}(t)\right) \cdot \left(1 - \bigwedge_{j=1}^{n} R_{B,j}^{(U)}(t)\right). \end{aligned} \tag{V.8}$$

B. Belief reliability formula for complex systems

For more complex system, such as an uncertain random k-out-of-n system, it is much harder to obtain the system belief reliability functions directly. In this paper, we assume the system only have two states. Therefore, at first, we do not consider the effect of time, then the uncertain random system can be regarded as a Boolean system. In this case, the system reliability formula proposed by Wen and Kang [37] is first adopted. It is noted that the formula can be easily extended to a time-variant situation by performing the formula at each time of the system lifetime.

Theorem V.3. (Wen and Kang[37]) *Assume that a Boolean system has a structure function f and contains random components with belief reliabilities $R_{B,i}^{(P)}(t), i=1,2,\cdots,m$ and uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j=1,2,\cdots,n$. Then the belief reliability of the system is*

$$R_{B,S}(t)=\sum_{(y_1,\cdots,y_m)\in\{0,1\}^m}\left(\prod_{i=1}^{m}\mu_i(y_i,t)\right)\cdot Z(y_1,y_2,\cdots,y_m,t), \tag{V.9}$$

where

$$Z(y_1,y_2,\cdots,y_m,t)=\begin{cases}\sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=1}\min\limits_{1\le j\le n}\nu_j(z_j,t), \\ \quad \text{if } \sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=1}\min\limits_{1\le j\le n}\nu_j(z_j,t)<0.5,\\ 1-\sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=0}\min\limits_{1\le j\le n}\nu_j(z_j,t),\\ \quad \text{if } \sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=1}\min\limits_{1\le j\le n}\nu_j(z_j,t)\ge 0.5,\end{cases}$$

$$\mu_j(y_i,t)=\begin{cases}R_{B,i}^{(P)}(t), & \text{if } y_i=1,\\ 1-R_{B,i}^{(P)}(t), & \text{if } y_i=0,\end{cases}(i=1,2,\cdots,m),$$

$$\nu_j(z_j,t)=\begin{cases}R_{B,i}^{(U)}(t), & \text{if } z_j=1,\\ 1-R_{B,i}^{(U)}(t), & \text{if } z_j=0,\end{cases}(j=1,2,\cdots,n).$$

C. Case study

Consider an apogee engine of a satellite proposed in [41]. The mission of the engine is to start at apogee and send the satellite into the synchronous orbit. To ensure the the success of the mission, the engine should not fail within the first 2 working hours. The engine is mainly consist of four parts: an ignition structure, an engine shell, a propellant grain and a nozzle. The ignition structure can be further decomposed into three kinds of components, namely, an igniter, two spark plugs (including one back-up spark plug) and some ignition composition. Among the components, the ignition composition and the nozzle are innovative products with few failure data, so they are modeled as uncertain components in this paper. The failure time distributions of the two components are obtained based on experts' empirical data. Since other components are mature product with a lot of failure time data, we model them as random components, whose failure time distributions are obtained through statistical method based on field or experimental failure data.

The working process of the apogee engine can be summarized as follows. First the igniter receives command and generates a pulse, then it ignites the ignition composition through the spark plugs. Later, the propellant grain is burned, generating a lot of gas. The gas will ejected from the nozzle to outside, thereby propel the satellite. The whole process takes place in the engine shell. It can be easily noticed that the engine will fail whenever a sort of component fail. Therefore, the reliability block diagram of the apogee engine can be represented by Fig. 1. The failure time distributions of these components are listed in Table I.

TABLE I
FAILURE TIME DISTRIBUTIONS OF COMPONENTS

No.	Component type	Failure time distribution
1-1,1-2,1-2'	Random	Exponential$(\lambda=10^{-2.5}h^{-1})$
1-3	Uncertain	$\mathcal{L}(150h,400h)$
2,3	Random	Exponential$(\lambda=5\times10^{-3.5}h^{-1})$
4	Uncertain	$\mathcal{L}(100h,500h)$

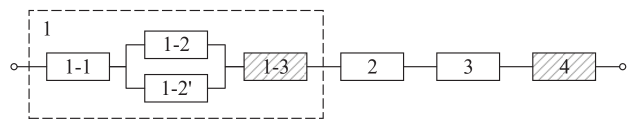

Fig. 1. Reliability block diagram of the apogee engine (1: ignition structure, 2: engine shell, 3: propellant grain, 4: nozzle, a: igniter, b: spark plug, b: back-up plug, c: ignition composition)

By merging the same type of components, we obtain two subsystems: a random subsystem containing all random components 1-1, 1-2, 1-2', 2 and 3, and an uncertain subsystem containing all uncertain components 1-3 and 4. We first calculate the belief reliability of the two subsystems.

$$\begin{aligned}R_{B,R}^{(P)}(t)&=\left[1-\left(1-R_{B,1-2}^{(P)}(t)\right)\left(1-R_{B,1-2'}^{(P)}(t)\right)\right]\cdot\\&\quad R_{B,1-1}^{(P)}(t)\cdot R_{B,2}^{(P)}(t)\cdot R_{B,3}^{(P)}(t)\\&=e^{-2\times10^{-2.5}t}\cdot\left(2e^{-10^{-2.5}t}-e^{-2\times10^{-2.5}t}\right)\\&=2e^{-3\times10^{-2.5}t}-e^{-4\times10^{-2.5}t},\end{aligned} \tag{V.10}$$

$$\begin{aligned}R_{B,U}^{(U)}(t)&=R_{B,1-3}^{(U)}(t)\wedge R_{B,4}^{(U)}(t)\\&=\begin{cases}1, & \text{if } t\le 100h,\\ \frac{600-t}{500}, & \text{if } 100h<t\le 200h,\\ \frac{400-t}{250}, & \text{if } 200h<t\le 400h,\\ 0, & \text{if } t>400h.\end{cases}\end{aligned} \tag{V.11}$$

Then the system can be regarded as a series system consisting of a random subsystem and an uncertain subsystem. According to Theorem V.1, we have

$$R_{B,S}(t)=R_{B,R}^{(P)}(t)\cdot R_{B,U}^{(U)}(t). \tag{V.12}$$

The system belief reliability function is illustrated in Fig.2. We can get the system reliability at $2h$ is $R_{B,S}(2h)=0.9874$. Using numerical methods, the MTTF of the system can be also calculated to be MTTF $=114.25h$. This means that we can have a good faith that the mission will be successful.

VI. CONCLUSION

In this paper, belief reliability metric is defined based on chance theory to measure reliability of uncertain random

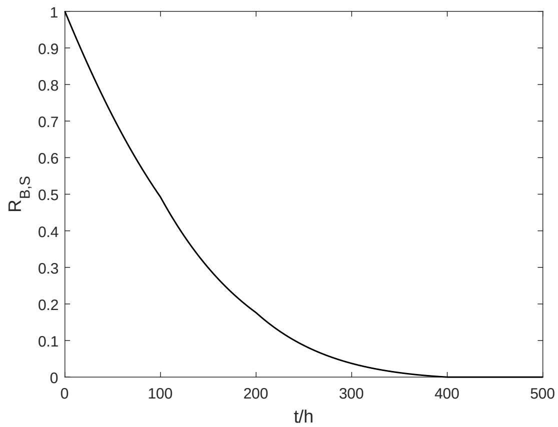

Fig. 2. System belief reliability of the apogee engine

systems affected by both aleatory and epistemic uncertainties. The developed metric can degenerate to the probability theory-based reliability metric or the uncertainty theory-based belief reliability metric. Some belief reliability indexes, including belief reliability distribution, MTTF, belief reliable life, are proposed on the basis of the belief reliability metric. In addition, this paper managed to propose some system belief reliability formulas. For different system configurations, different belief reliability formula can be used. A simple case study is given to further illustrated the proposed metric and formulas.

In conclusion, the contributions of this paper are as follows:

(1) The definition and concept of *belief reliability* is expanded based on chance theory considering both aleatory and epistemic uncertainties.
(2) The belief reliability metric and its physical meaning is first interpreted in detail from the view of failure time, performance margin and function level.
(3) Some new belief reliability indexes are proposed based on the new definition of belief reliability.
(4) Several new system belief reliability formulas are developed to analyze the belief reliability of uncertain random systems.

The future works may focus on the reliability modeling, analysis methods, etc. One of the most interesting issues is to obtain the belief reliability of a product through its physical model of performance margin, as briefly elaborated in this paper. Another important and interesting problem is to give a belief reliability evaluation method of muti-state systems.

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Quantifying Epistemic Uncertainty in Belief Reliability Based on Technology Readiness Levels

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Abstract

Belief reliability considers the influences of design margin, aleatory uncertainty and epistemic uncertainty, which gives a more comprehensive evaluation on reliability. The quantification of epistemic uncertainty plays a pivotal role in belief reliability analysis. In this paper, we establish a model to quantify epistemic uncertainty based on technology readiness levels (TRLs) which illustrate the state of technology in terms of maturity. First, a general evaluating process for TRLs is developed. Based on this, the concepts of quality of technology readiness condition and technology readiness score are first proposed to comprehensively measure the TRL. Then, the epistemic uncertainty is quantified through a mathematical model which integrating the effect of TRL and the technology readiness score. Finally, a case study is applied on a quad redundant servo system to demonstrate the proposed model.

Keywords: Uncertainty Modeling, Epistemic Uncertainty, Technology Readiness Levels

1. Introduction

Traditionally, reliability is defined to be the probability that a component or system will perform a required function for a given period of time when used under stated operating conditions [1]. It is the probability of non-failure over time and then the failure time has been adopted as original data to estimate reliability by statistical methods [2]. Model-based approaches have been then proposed to predict reliability when the failure data is s-carce, which includes physics-of-failure methods [3, 4], structural reliability methods [5, 6], etc. These methods use a deterministic model to describe the failure behavior of a component or a system and uncertainties are taken into consideration by using random variables in these deterministic models. Such type of uncertainty could be referred to as aleatory uncertainty as it is the inherent property in the physics behavior of the system. In contrast with aleatory uncertainty, there also exits epistemic uncertainty influencing reliability assessment. Epistemic uncertainty derives from a lack of knowledge about the appropriate value to use for a quantity that is assumed to have a fixed value in the context of the particular analysis or the adoption of deterministic models [7, 8].

Regarding the impact of epistemic uncertainty, Zeng et al. [9] proposed belief reliability which considers the influences of design margin, aleatory uncertainty and epistemic uncertainty. In that framework, epistemic uncertainty was quantified based on the state of knowledge by evaluating the commonly used epistemic-uncertainty-related engineering activities, i.e., Failure Mode, Effect and Criticality Analysis (FMECA), Failure Reporting, Analysis and Corrective Action System (FRACAS), Reliability Enhancement Test (RET), Reliability Growth Test (RGT) and Reliability Simulation Test (RST) [9]. These reliability engineering activities could help designers better understand potential failure modes and mechanisms. In other words, these reliability engineering activities were employed to quantify the state of knowledge in terms of failure.

In this paper, epistemic uncertainty is quantified by technology readiness levels (TRLs). TRLs were first proposed by the National Aeronautics and Space Administration (NASA) as a discipline-independent programmatic figure of merit to allow more effective assessment and communication regarding the maturity of new technologies [10]. The TRL system was defined as a seven-level one at 1989 [11] and then in 1995, the TRL scale was further strengthened by the articulation of the definitions of each level along with examples [12]. Over the last three decades, the TRL system has been tailored to and adopted by various industrial and government organizations,such as U.S. Department of Energy, U.S. Government Accountability Office and International Organization for Standardization [13–15]. TRLs have been used to evaluate the maturity of hardware and software technologies critical to the performance of a large system or the fulfillment of the key objectives of an acquisition program [10] and have provided a means of managing risks and making better decisions in various fields, such as gas[16], aviation industry[17], Terahertz[18], recycling technology[19] and bioprinting [20]. In general, the TRL system describes a typical product development and deployment path. This metric classifies the maturity status of a technology staring from basic principles all way up to systems proven through successful mission operations. The initial TRL is usually defined as basic discipline research. At this level, only some basic concepts or principles have been obtained by researchers. Thus, epistemic uncertainty is formidable and employing technology at this level may lead to tremendous risk. The final TRL is generally referred to system test. At this level, actual system has been proved through successful system and/or mission operations, which could be inferred that the researchers have a better command of the system and there is less epistemic uncertainty than that at the initial level. Therefore, a quantification model based on TRLs is proposed to quantify epistemic uncertainty.

The remainder of this paper is organized as follows. In Section 2, a general evaluating framework of TRLs is developed by summarizing previous work. An approach is proposed to quantify technology readiness in Section 3. Epistemic uncertainty quantification model and its properties are discussed in Section 4. A quad servo system is subsequently adopted as a case to demonstrate the developed methods in Section 5. Finally, the paper is concluded in Section 6.

2. General Evaluating Framework of TRLs

In this section, a general evaluating framework of TRLs is developed by summarizing existing literatures.

2.1. Classification and definition of TRLs

Clarifying the classification and definition of each level is the first thing to do when evaluating readiness level of a technology. This work could base on either general definitions from existing standards or appropriative classifications for specific technologies. It is helpful to refer to extant definitions to indicate the maturity of a given technology when the range of definitions is appropriate. For example, Redo-Sanchez et al. [18] assessed the maturity of Terahertz technology based on the definitions published by NASA [12]. Liu and Fan [21] adopted the classifications released by U.S. DOE to evaluated technology readiness of Small Modular Reactor (SMR) designs. On the other hand, there are cases beyond the scope of existing standards, where specified classifications and definitions of TRLs are proposed to satisfy practical demands. For instance, Papafotiou et al. [22] proposed specialized descriptions of TRLs for model predictive control technology in power electronics and Carmack et al. [23] firstly provided definitions of TRLs for nuclear fuel technology to assess the maturity of nuclear fuels and materials under development.

2.2. Determination of technology readiness conditions

After clarifying the classification and definition of TRLs, the next step is to determine detailed technology readiness conditions for each TRL. Generally, these conditions could be classified into three types, technology conditions, manufacture conditions and management conditions.

1. Technology conditions consist of design conditions and verification conditions.
    (a) Design conditions are referred to as research and design work completed in the development stage, which include acquaintance with application requirements and operational environments, confirmation of hypotheses and scientific principles, determination of technical properties and compilations of technical sources.
    (b) Verification conditions are referred to as verification issues based on the results of research and development, which are comprised of objects, environments, results and key projects.
2. Manufacture conditions involve craft designs, manufacturing processes and processing facilities during the period of trial-manufacture or manufacture.
3. Management conditions mainly include customer relationship managements, risk managements and cost managements.

It should be noted that each level contains different numbers of technology conditions, manufacture conditions and management conditions and it is possible to not include all three types of conditions. For example, TRL 1 is referred to as basic principles observed and reported[14]. It is likely to only contain technology conditions at TRL 1.

2.3. Judgment on TRLs

The following step is to judge the current TRL of candidate technology as shown in Fig.1 [24]. The judgment process could be divided into two steps, initial judgment and further judgment. During the period of initial judgment, it is necessary to roughly assess the TRL of candidate technology based on the definition of TRLs and the research and development phase. Further judgment is conducted based on the conclusion of initial judgment. If all conditions of initial TRL N are satisfied, then evaluate the conditions of a higher level until all the conditions of level N are satisfied while the conditions of level $N+1$ are not. If all conditions of initial TRL N are not satisfied, then evaluate the conditions of a lower level, i.e. $N-1$ level, until all conditions of that level are satisfied or the conditions of TRL 1 are not all satisfied. Eventually, the conclusion could be made from the above analysis.

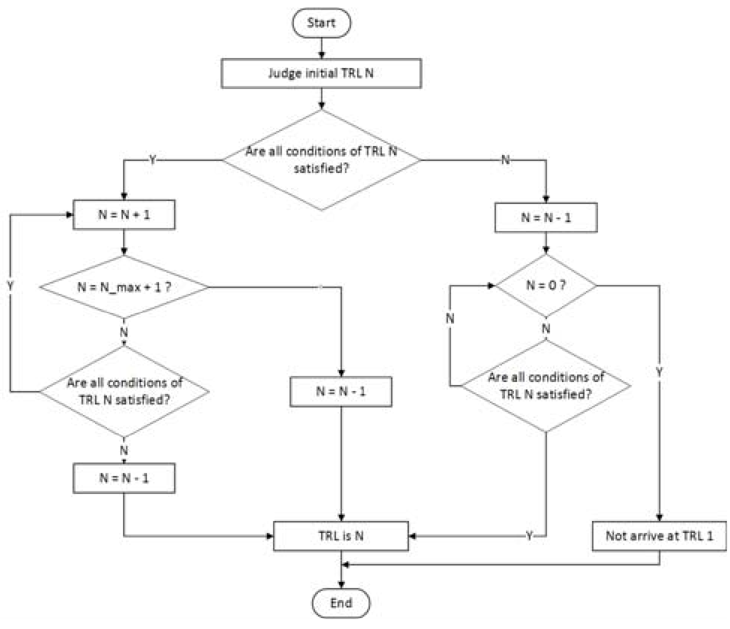

Figure 1: Process diagram of judgment on TRLs

2.4. General evaluating framework

The general evaluating framework of TRLs contains three main steps, classification and definition of TRLs, determination of technology readiness conditions of each TRL and judgment of current TRL N. The whole process is summarized in Fig.2. It can be obtained that the output of the framework is current TRL N and all conditions of this TRL.

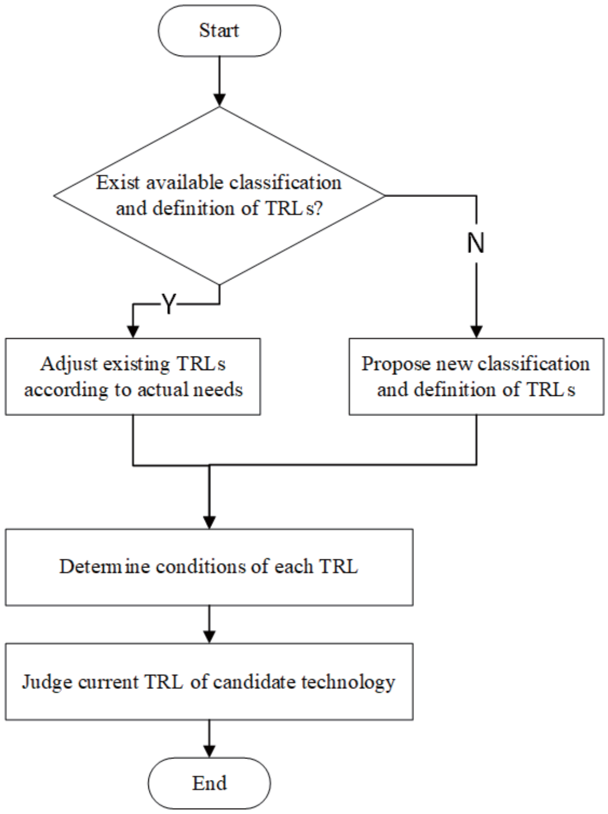

Figure 2: General evaluating framework of TRLs

3. Quantification of technology readiness

In this section, an approach is proposed to quantify technology readiness based the output of the general evaluating framework. Two concepts, quality of technology readiness condition and technology readiness score, are firstly defined to give a comprehensive description of technology readiness. Quality of technology readiness condition measures the performance of technology readiness conditions. Technology readiness score is a quantity to represent the quantification result of technology readiness and is calculated by quality of technology readiness conditions in conjunction with weights of experts, relative contributions and relative importance.

3.1. Basic definitions

The execution of the same technology readiness condition may differ from each other in different application scenarios. For example, the condition of TRL 9 of NASA is that the final product is successfully operated in an actual mission [12]. We assume that there are two cases. The success number of the two cases are 1 and 100. The two cases have both reached the requirement of TRL 9 but the executions are obviously different. Therefore, it is necessary to measure the performance of each condition. Quality of technology readiness condition is proposed for this purpose.

Definition 1. Quality of technology readiness condition measures to what extend a technology readiness condition is well executed. It is denoted by Q, $Q \in (0, 1]$.

It is noted that the larger the quality Q is, the better the condition is executed. When Q goes to zero, the condition is believed to be just completed. On the other hand, the execution is believed to be well enough when Q reaches one.

Definition 2. Technology readiness score measures comprehensive quality of one technology readiness level.

Let Q_{ijk} be the quality of j-th technology readiness condition of i-th type evaluated by k-th expert, ω_i be the weights of relative importance of condition type, ω_{ij} be the weights of relative contributions of technology readiness condition and ω_{ijk} be the weights of experts, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, m_i$, $k = 1, 2, \cdots, n$, then technology readiness score S is calculated by:

$$S = \sum_{i=1}^{p} \omega_i \left(\sum_{j=1}^{m_i} \omega_{ij} \left(\sum_{k=1}^{n} \omega_{ijk} Q_{ijk} \right) \right), \tag{1}$$

where $\sum_{i=1}^{p} \omega_i = 1$, $\sum_{j=1}^{m_i} \omega_{ij} = 1$ and $\sum_{k=1}^{n} \omega_{ijk} = 1$.

It is noted that the larger the score S is, the better the quality of TRL is. When S goes to zero, the TRL is believed to be just arrived at. On the other hand, the TRL is believed to be executed well enough when S reaches one.

Technology readiness score is essentially determined by quality of technology readiness conditions. However, different quantification methods of weights of experts, relative contributions and relative importance also have a pivotal influence on the result of technology readiness score. The quantification methods of these three types of weights will be discussed in the following subsections.

3.2. Determination of weights of experts

Assume that there are n experts participate in the evaluation of quality of technology readiness condition. Then there will be n crisp numbers for

j-th technology readiness condition of i-th type, $Q_{ij1},\ Q_{ij2},\cdots,Q_{ijn}$, which reflects the opinions to the same object from different experts. It is believed to be ideal if these numbers equal to each other. However, these numbers are always different from each other. A weight determination method based on the distance is adopted here to get a comprehensive result from these scattered numbers.

Let $\overline{Q}_{ij}=\frac{1}{n}\sum_{k=1}^{n}Q_{ijk},\ \overline{Q}_{ij}\in(0,1]$ be the mean value of quality of j-th condition of i-th type. Then the distance d_{ijk} is defined as the absolute value between Q_{ijk} and $\overline{Q}_{ij}$ and it is calculated by:

$$d_{ijk}=|Q_{ijk}-\overline{Q}_{ij}|,\ d_{ijk}\in[0,1). \tag{2}$$

Subsequently, ω^*_{ijk} is determined by the distance d_{ijk} and acceptable threshold λ_E.

$$\omega^*_{ijk}=\begin{cases}\dfrac{1}{\varepsilon}, & if\ d_{ijk}=0,\\ \dfrac{1}{d_{ijk}}, & if\ 0<d_{ijk}\le\lambda_E,\\ \dfrac{1}{10d_{ijk}}, & if\ d_{ijk}>\lambda_E,\end{cases} \tag{3}$$

where ε is the calculation accuracy.

λ_E is the acceptable range of score discrepancies of expert opinion. The larger the λ_E is, the higher the tolerance of score discrepancies is. An expert's opinion is distinguished from other experts' when the distance is larger than acceptable threshold. Therefore, this expert's opinion should be less important by the principle of majority compliance and the corresponding weight should be relatively smaller.

Finally, the weights of experts is calculated from Eq.(4).

$$\omega_{ijk}=\frac{\omega^*_{ijk}}{\sum_{k=1}^{n}\omega^*_{ijk}}. \tag{4}$$

3.3. Determination of weights of relative contributions

For convenience, let $Q_{ij}=\sum_{k=i}^{n}\omega_{ijk}Q_{ijk}$ be the quality of j-th technology readiness condition of i-th type.

Suppose that there are m_i quality of technology readiness conditions for i-th type, $Q_{i1},Q_{i2},\cdots,Q_{im_i}$. It is intuitive that excellent quality of condition can help to reduce the epistemic uncertainty. Therefore, the better quality of condition is regarded to be more important.

Based on this principle, the weights of relative contributions of technology readiness condition, denoted by ω_{ij}, is determined as follows:

$$\begin{cases} \omega_{ij}^* & = e^{-\max\{\lambda_L - Q_{ij}, 0\}}, \\ \omega_{ij} & = \dfrac{\omega_{ij}^*}{\sum_{j=1}^{m_i} \omega_{ij}^*}, \end{cases} \tag{5}$$

where λ_L is acceptable quality of technology readiness condition.

Since it almost has no contribution to reduce epistemic uncertainty when Q_{ij} is near to zero, the corresponding weight ω_{ij}^* is relatively low. ω_{ij}^* increases as the value of Q_{ij} increases. λ_L is the minimum value over which there is no more decrement of epistemic uncertainty when Q_{ij} is increasing. Therefore, ω_{ij}^* equal to 1 when Q_{ij} is over λ_L. Finally, the weights of relative contributions ω_{ij} is calculated by ω_{ij}^*.

3.4. Determination of weights of relative importance of condition type

For convenience, let $Q_i = \sum_{j=1}^{m_i} \omega_{ij} Q_{ij}$ be the quality of i-th type technology readiness condition.

Suppose that there are p quality of technology readiness types, $Q_1, Q_2, \cdots, Q_p$. The weights of them are determined by pairwise comparison. Construct a pairwise comparison matirx $A_{p\times p}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{pmatrix} \tag{6}$$

where $a_{ij} = 1/a_{ji}$.

a_{ij} are determined by using relative scale measurement shown in Table 1. The pair-wise comparisons are done in terms of which type dominates the other.

Having made the pair-wise comparisons the consistency is determined by using the eigenvalue λ_{max} to calculate the consistency index, CI as follows: $CI = (\lambda_{max} - p)/(p-1)$, where p is the matrix size.

Judgment consistency can be checked by taking the consistency ratio (CR) of CI with the appropriate value in Table 2. The CR is acceptable, if it does not exceed 0.10. Otherwise, the judgment matrix is inconsistent. To obtain a consistent matrix, judgments should be reviewed and improved.

Table 1: Pair-wise comparison scale for technology readiness types

Numerical rating	Verbal judgments
9	Extremely important
8	Very strongly to extremely important
7	Very strongly important
6	Strongly to Very strongly
5	Strongly important
4	Moderately to strongly
3	Moderately important
2	Equally to moderately
1	Equally important

Table 2: Average random consistency (RI)

Size of matrix	1	2	3	4	5	6	7	8	9	10
Random consistency	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

Then $(\omega_1^*, \omega_2^*, \cdots, \omega_p^*)$ is the eigenvector of λ_{max}. The weights of relative importance of condition type is calculated by:

$$\omega_i = \frac{\omega_i^*}{\sum_{i=1}^{p} \omega_i^*}. \tag{7}$$

4. Epistemic Uncertainty Quantification Model

In this section, the relationship among epistemic uncertainty, current technology readiness level and technology readiness score is displayed in the epistemic uncertainty quantification model.

Definition 3. Epistemic uncertainty quantification model is a function aiming for quantifying epistemic uncertainty by current technology readiness level and technology readiness score. Let E be epistemic uncertainty, N be current technology readiness level and S be technology readiness score. Then epistemic uncertainty quantification model is:

$$E(N, S) = \frac{(1-S)}{N \times S^2}, N \in \{1, 2, \cdots, N_{max}\},\ S \in\ (0, 1]. \tag{8}$$

Theorem 1.

(1) E is a monotone decreasing function of S,

(2) The absolute value of slope of E is a monotone decreasing function of S,

(3) $\lim_{S \to 0} E = \infty$,

(4) $E(N, 1) = 0$.

Proof:

(1) As $S \in (0, 1]$ and N is a positive integer, we have:

$$\frac{\partial E}{\partial S} = \frac{-S^2 - 2(1-S)S}{N \times S^4} = \frac{S-2}{N \times S^3} < 0.$$

Therefore, E is a monotone decreasing function of S.

(2) As $S \in (0, 1]$ and N is a positive integer, we have:

$$\frac{\partial^2 E}{\partial S^2} = \frac{S^3 - 3(S-2)S^2}{N \times S^6} = \frac{6 - 2S}{N \times S^4} > 0.$$

which shows that the tendency is monotone.

Besides,

$$\lim_{S \to 0} \frac{\partial E}{\partial S} = -\infty, \ \frac{\partial E}{\partial S}(N, 1) = -\frac{1}{N},$$

Thus, the slope rises from $-\infty$ to $-1/N$ as S increases from 0 to 1, which means that the absolute value of slope of E decreases from $+\infty$ to $1/N$. Therefore,the absolute value of slope of E is a monotone decreasing function of S.

(3) It is obvious.

(4) It is obvious.

It could be learned from theorem 1.(1) that epistemic uncertainty decreases as technology readiness score becomes higher, which could be seen in Fig.3. Since technology readiness score increases when quality of TRL conditions improve, the higher technology readiness score reflects the better capability of knowledge on the technology. Therefore, it is reasonable that epistemic uncertainty is a monotone decreasing function of technology readiness score. Besides, epistemic uncertainty falls more sharply at the initial levels than that at the senior levels. An explanation for this is the designated conditions of the initial levels are easier to achieve than those of the senior levels. On the other hand, the tolerance to epistemic uncertainty at the initial levels is higher than that at the senior levels.

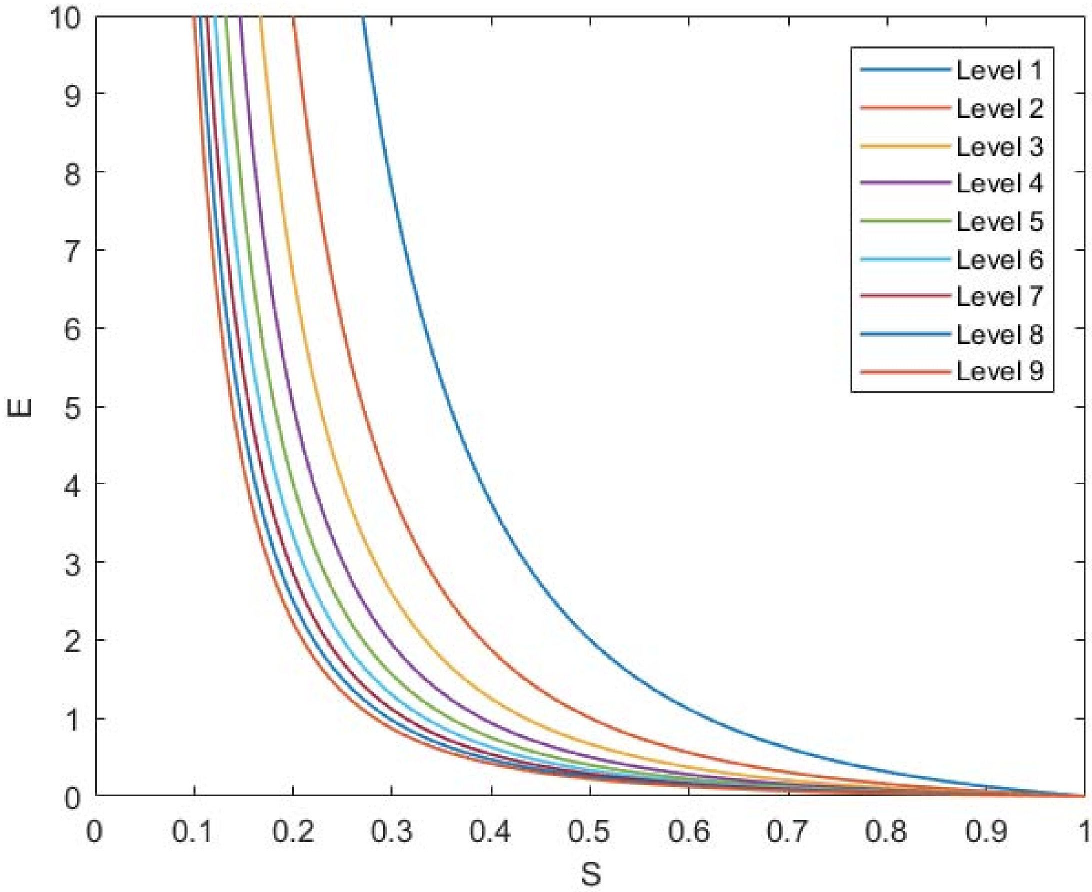

Figure 3: Epistemic uncertainty quantification model with $N \in \{1, 2, \cdots, 9\}$

It could be learned from theorem 1.(2) that the absolute value of slope of e reduces with S. Since high technology readiness score requires excellent quality of each TRL condition. Therefore, it is a reasonable explanation that the absolute value of slope of epistemic uncertainty is decreasing as the difficulty of getting the increment of S is increasing.

It should also be noted that when technology readiness score S goes to zero, epistemic uncertainty goes to infinite. This is because we almost know nothing about the current technology when S is close to zero. Therefore, epistemic uncertainty is regarded to be infinite. Besides, epistemic uncertainty goes to zero when technology readiness score achieves one. An reasonable assumption is that the epistemic uncertainty disappears when every condition is impossible to do better.

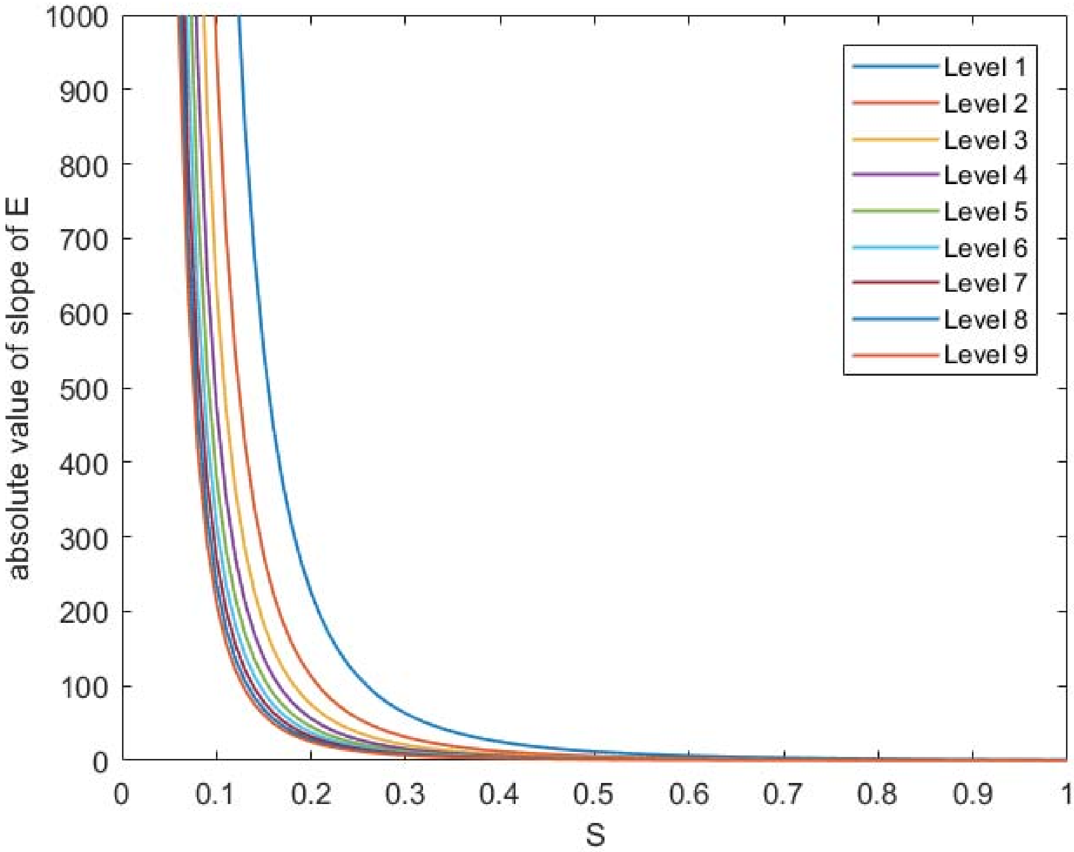

Figure 4: Absolute value of slope of E is reducing with S. with $N \in \{1, 2, \cdots, 9\}$

5. Case Study

In this section, we apply the developed method to quantify epistemic uncertainty of a quad redundant servo system. The quad redundant servo system mainly includes three parts, i.e., main control unit, power drive unit and sensor unit. Each group of the three units constitutes one channel of servo system as shown in Fig.5.

The definitions of TRLs for the quad redundant servo system is based on GJB 7688-2012 [25] and specific conditions are tailored considering the practical scenarios. The quad redundant servo system has been evaluated at TRL 6 and technology readiness conditions are shown in Table 3.

As shown in the Table 3, there are three types of technology readiness conditions, including 10 technology conditions, 5 manufacture conditions and 5 management conditions.

Five experienced experts involved in the evaluation of quality of technol-

Table 3: Specific conditions of TRL 6

No.	Type	Contents
1	Technology	Find out final operational environment.
2	Technology	Complete simulation tests on performance of expected system in operational environment.
3	Technology	Complete the acceptance test.
4	Technology	Complete the field test on the prototype.
5	Technology	Integration of prototype approaches the requirement of final system.
6	Technology	Certificate the project feasibility of technology.
7	Technology	Analyze effects of the differences between the text environment and operational environment.
8	Technology	Complete most of the design drafts
9	Technology	Complete the final copy of technology report.
10	Technology	Propose a patent application
11	Manufacture	Determine the degree of quality and reliability.
12	Manufacture	Collect actual data on reliability, maintainability and supportability.
13	Manufacture	Determine the investment on manufacturing process and equipment.
14	Manufacture	Roughly determine the specification on critical manufactory process.
15	Manufacture	Complete demonstrate experiments of the production.
16	Management	Propose the target of expense control.
17	Management	Propose the plan of system engineering management.
18	Management	Determine the milestones of the project.
19	Management	Draft the production plans.
20	Management	Determine the formal requirements documents.

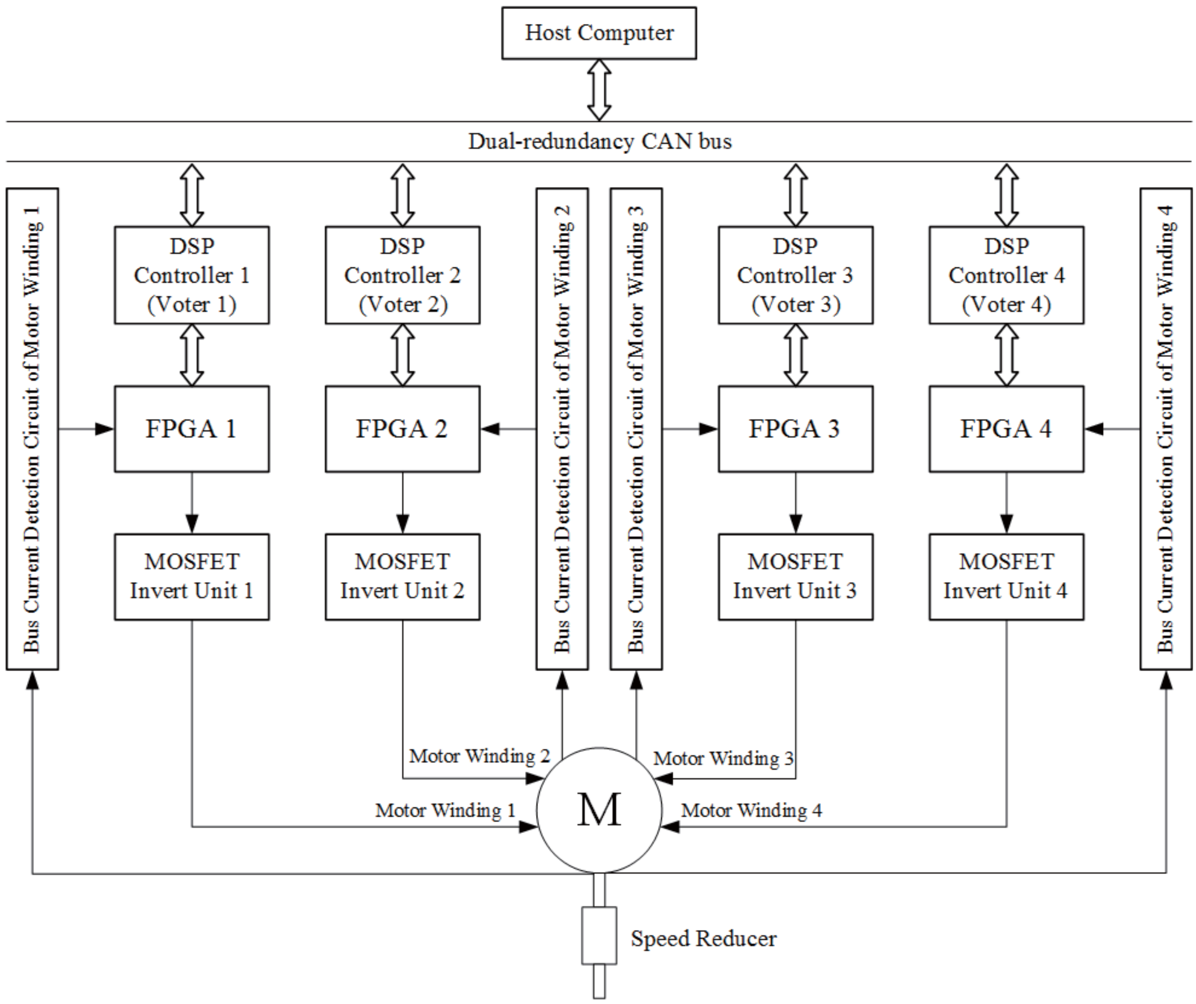

Figure 5: Quad redundant servo system

ogy readiness condition and the results are shown in Table 4. Expert weights are calculated by Eq.(3) and Eq.(4) with $\varepsilon = 0.01$ and $\lambda_E = 0.15$ and the results are shown in Table 5. Then Q_{ij} could be obtained by the above results. Weights of relative contributions ω_{ij} are dertemined by Eq(5) with $\lambda_L = 0.75$ based on Q_{ij}. The value of Q_{ij} and ω_{ij} are also displayed in Table 5. The quality of technology conditions Q_1, quality of manufacture conditions Q_2 and quality of management conditions Q_3 could be acquired based on Q_{ij} and ω_{ij}. Then we have $Q_1 = 0.60,\ Q_2 = 0.52,\ Q_3 = 0.62$.

The pairwise comparison matrix $A_{3\times3}$ is given as follows:

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 1/5 & 1 & 1/2 \\ 1/3 & 3 & 1 \end{pmatrix}. \tag{9}$$

Then we have $\lambda_{max} = 3$, $CI = 0$ and $CR = 0$ which satisfies $CR < 0.10$. The eigenvector of λ_{max} is $(\omega_1^*, \omega_2^*, \omega_3^*) = (0.93, 0.17, 0.33)$. Therefore, we have $(\omega_1, \omega_2, \omega_3) = (0.65, 0.12, 0.23)$. Finally, the technology readiness score is 0.60 and epistemic uncertainty E is 0.19 by Eq.(8) with $N = 6$.

Table 4: Expert scores

No.	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5
1	0.6	0.6	0.9	0.5	0.4
2	0.8	0.5	0.7	0.8	0.6
3	0.2	0.6	0.5	0.5	0.8
4	0.4	0.6	0.4	0.3	0.8
5	0.7	0.4	0.8	0.6	0.6
6	0.7	0.3	0.9	0.9	0.8
7	0.2	0.6	0.5	0.8	0.1
8	0.2	0.6	0.8	0.7	0.6
9	0.9	0.8	0.7	0.5	0.9
10	0.3	0.8	0.4	0.3	0.2
11	0.5	0.9	0.4	0.8	0.1
12	0.5	0.5	0.3	0.4	0.2
13	0.4	0.7	0.8	0.4	0.9
14	0.4	0.3	0.1	0.5	0.3
15	0.6	0.2	0.8	0.8	0.6
16	0.9	0.7	0.9	0.5	0.7
17	0.6	0.7	0.6	0.8	0.8
18	0.3	0.9	0.5	0.6	0.6
19	0.7	0.9	0.1	0.8	0.5
20	0.5	0.3	0.5	0.5	0.6

6. Conclusions

In this paper, we proposed a model to quantify epistemic uncertainty based on TRLs. A general evaluating framework of TRLs was developed by summarizing previous literature. The framework was composed of three main steps, classification and definition of TRLs, determination of technology readiness conditions of each TRL and judgment of current TRL N. The output of this process was current TRL of candidate technology along with technology readiness conditions of this level. A method was then proposed to

Table 5: Expert weights and quality of technology condition

$k=$	1	2	3	4	5				
w_{11k}	0.47	0.47	0.00	0.05	0.00	Q_{11}	0.59	w_{11}	0.10
w_{12k}	0.10	0.01	0.63	0.10	0.16	Q_{12}	0.70	w_{12}	0.11
w_{13k}	0.00	0.11	0.44	0.44	0.00	Q_{13}	0.51	w_{13}	0.09
w_{14k}	0.32	0.32	0.32	0.02	0.01	Q_{14}	0.46	w_{14}	0.09
w_{15k}	0.11	0.00	0.00	0.44	0.44	Q_{15}	0.61	w_{15}	0.10
w_{16k}	0.78	0.00	0.01	0.01	0.20	Q_{16}	0.72	w_{16}	0.11
w_{17k}	0.02	0.03	0.91	0.02	0.02	Q_{17}	0.50	w_{17}	0.09
w_{18k}	0.00	0.46	0.00	0.08	0.46	Q_{18}	0.61	w_{18}	0.10
w_{19k}	0.13	0.44	0.30	0.01	0.13	Q_{19}	0.80	w_{19}	0.12
w_{110k}	0.08	0.00	0.83	0.08	0.00	Q_{110}	0.38	w_{110}	0.08
w_{21k}	0.76	0.01	0.22	0.01	0.01	Q_{21}	0.49	w_{21}	0.20
w_{22k}	0.10	0.10	0.16	0.63	0.01	Q_{22}	0.40	w_{22}	0.18
w_{23k}	0.02	0.90	0.03	0.02	0.02	Q_{23}	0.69	w_{23}	0.24
w_{24k}	0.11	0.44	0.00	0.00	0.44	Q_{24}	0.31	w_{24}	0.16
w_{25k}	0.50	0.00	0.00	0.00	0.50	Q_{25}	0.60	w_{25}	0.22
w_{31k}	0.01	0.48	0.01	0.01	0.48	Q_{31}	0.70	w_{31}	0.22
w_{32k}	0.07	0.71	0.07	0.07	0.07	Q_{32}	0.69	w_{32}	0.21
w_{33k}	0.00	0.00	0.11	0.44	0.44	Q_{33}	0.58	w_{33}	0.19
w_{34k}	0.48	0.02	0.01	0.02	0.48	Q_{34}	0.61	w_{34}	0.20
w_{35k}	0.31	0.00	0.31	0.31	0.05	Q_{35}	0.50	w_{35}	0.18

quantify technology readiness based on the output of the evaluating framework. Quality of technology readiness condition was first defined to measure the performance of technology readiness condition. Then technology readiness score was calculated by quality of technology readiness condition in conjunction with weights of experts, weights of relative contributions and weights of relative importance. Epistemic uncertainty quantification model was then developed as a function to describe the relationship of epistemic uncertainty, current TRL N and technology readiness condition. Four theorems were discussed to illustrate the properties of epistemic uncertainty quantification model. A case study was applied to demonstrate the proposed model.

Compared to the existing method, epistemic uncertainty quantification model by TRLs expands the approaches to evaluate the impact of epistemic uncertainty in belief reliability. However, current model only considers the effect of a single unit. Further study will investigate the effect of system readiness level on epistemic uncertainties.

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Modeling Accelerated Degradation Data Based on the Uncertain Process

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***Abstract*—Accelerated degradation testing (ADT) aids the reliability and lifetime evaluations for highly reliable products. In engineering applications, the number of test items are generally small due to finance or testing resources constraints, which leads to the rare knowledge to evaluate reliability and lifetime. Consequently, the epistemic uncertainty is embedded in ADT data and the large-sample based probability theory is not appropriate any more. To solve this problem, in this paper, we introduce the uncertainty theory, which is a theory different from the probability theory, to account for such uncertainty due to small samples and build up a framework of ADT modeling. In this framework, an uncertain accelerated degradation model (UADM) is firstly proposed based on the arithmetic Liu process. Then, the uncertain statistics for parameter estimations is presented correspondingly, which is completely constructed on objectively observed ADT data. An application case and a simulation case are used to illustrate the proposed methodology. With further comparisons with the Wiener process based accelerated degradation model (WADM) and the Bayesian-WADM (B-WADM), the sensitivities of these models to sample sizes are explored, and the results show that the proposed methodology is superior to the other two probability-based models under the small sample size.**

***Index Terms*—Accelerated degradation testing, epistemic uncertainty, uncertainty theory, uncertain process, belief reliability**

I. Introduction

PRODUCT reliability contributes to quality and competitiveness to commercial enterprises. Hence, sufficient efforts have been devoted to the evaluation of product reliability and lifetime before releasing them to the market. For highly reliable and long lifespan products, like lithium-ion cells with the lifetime of several years, the corresponding reliability tests are rather time-consuming and inefficient. Thus, accelerated degradation testing (ADT) has been introduced [1]–[3]. Through more severe test conditions, the degradation process for products will be accelerated to obtain performance characteristics under limited time and financial constraints.

In general, there are mainly two kinds of models used for ADT modeling based on probability theory, which are degradation-path models [4] and stochastic process models [5], e.g. Wiener process [6], [7], Gamma process [8], inverse Gaussian process [9]. According to the law of large numbers, these probability theory based models are suitable for the situations where there are a large amount of samples so that the estimated probability distribution function is close enough to the long-run cumulative frequency to present the inherent aleatory uncertainty. However, in real engineering applications, there are usually only a few samples used in ADT because the tested samples are generally expensive and the corresponding testing resources are limited. Thus, only partial information can be obtained after the tests, which causes a lack of knowledge on recognizing the population and then leads to the epistemic uncertainty on conducting reliability evaluations. In such situations, the probability theory based models are not appropriated because only a few samples are available and the law of large numbers is no longer valid. For example, if the ADT data obtained from the small number of test samples lie in the statistical mean of the population, the reliability evaluation results will be acceptable. However, if the ADT data deviate from the mean value and even locates at the tail of the population, which is the most possibly occurred situation, the reliability evaluation results will be significantly conservative or radical. Therefore, when constructing a model for the ADT data, it is necessary to consider the epistemic uncertainty caused by small samples.

To quantify the epistemic uncertainty due to small samples, subjective information such as the belief degrees are usually considered to combine with the probability theory, namely imprecise probability method [10]. Bayesian method [11], interval analysis [12] and fuzzy probability theory [10], [13] are the three most commonly used imprecise probability methods. In Bayesian method, subjective information are treated as subjective probability by assigning priors to the parameters, so the epistemic uncertainty could be quantified. For instance, to quantify the epistemic uncertainty, Peng et al. [14] proposed a Bayesian analysis method for degradation modeling with the inverse Gaussian process; then, they [15] put forward a general Bayesian method to model the degradation process with time-varying degradation rates, in which the priors can be non-informative or informative. Both the simulation and case study results of the two papers show that the different priors exert a significant effect on the final evaluation results. As for the interval analysis and fuzzy probability theory, they quantify the epistemic uncertainty by assigning intervals or fuzzy variables with membership functions to the parameters, respectively. Details about the methodologies in this topic can be referred to [16]–[20].

Although the field of ADT modeling has been well developed for decades, there are still some problems in the

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existing ADT models. First of all, most of the existing ADT models are the probability theory based models. They are unappropriated for the situation where there exists epistemic uncertainty due to small samples, because small samples make the estimated probability distribution function not close enough to the long-run cumulative frequency and could lead to unacceptable evaluation results. Secondly, although there are many imprecise probability methods, there remains some problems. On the one hand, these methods still need to combine with the probability theory. On the other hand, these methods quantify the epistemic uncertainty by subjective measures. For instance, in Bayesian theory, the information used for estimating posteriors originate from two parts: priors and testing information. Because of the small samples in ADT, the testing information are quite limited, which makes the posteriors highly depend on the priors. However, these priors are usually directly predetermined by engineers in real applications. Different engineers will contribute different subjective opinions, which make great impacts to the priors. Similar problem occurs to interval analysis and fuzzy probability theory. Thirdly, if there are several model parameters described by priors, intervals or fuzzy variables, the obtained reliability and lifetime evaluation intervals might be too broad to provide sufficient support for decision-making. In addition, using these imprecise probability methods might lead to a huge amount of computation. For instance, in the Bayesian theory, the complex priors make it difficult to get an explicit expression of the reliability function. So simulation approaches are usually adopted to obtain the approximations, which will lead to a large amount of calculation. The interval analysis and fuzzy probability theory have the similar situations.

When the sample size is very small and the probability theory is not appropriate anymore, the uncertainty theory utilizes belief degrees to describe the chance that an event occurs [21]. Thus, we introduce the uncertainty theory to the field of ADT modeling to address the aforementioned problems. The uncertainty theory was proposed by Liu [22] and has been introduced to fields like reliability analysis [23]–[25], risk analysis [26], supply chain [27], etc. To describe the dynamic uncertain event, Liu [28] proposed an uncertain process (also known as the Liu process), which is a sequence of uncertain variables indexed by time, presented the uncertainty distribution to describe uncertain process [29], and provided the extreme value theorem and the uncertainty distribution of the first hitting time (FHT) for independent increment uncertain process [30].

In this paper, the uncertainty theory is used to quantify the epistemic uncertainty caused by the small sample problem in ADT data. Considering the dynamic phenomena of degradation process, the uncertain process is utilized to describe the deterioration of products and construct an uncertain accelerated degradation model. Based on the proposed model, this paper also derives the implicit expression of the reliability and lifetime distributions, and proposes the uncertain statistics method for parameter estimations, in which the epistemic uncertainty is quantified by objective measures. The rest of this paper is organized as follows. Section II gives some preliminaries about the uncertainty theory. Section III presents the methodology of uncertain accelerated degradation modeling with the uncertainty distribution of the first hitting time (FHT) and corresponding uncertain statistics method. Section IV conducts the case study and the sensitivity analysis. Section V concludes the paper.

II. Uncertainty Theory

Let Γ be a nonempty set and $\mathcal{L}$ be a σ - algebra over Γ. $(\Gamma, \mathcal{L})$ presents a measurable space. Each element Λ in $\mathcal{L}$ is a measurable set. A set function $\mathcal{M}$ from $\mathcal{L}$ to [0, 1] is called an uncertain measure if it satisfies the normality axiom, the duality axiom, the subadditivity axiom [22], and the product measure axiom [31].

Definition 1. [22] *An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel Set B of real numbers, i.e.*

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}. \tag{1}$$

Definition 2. [22] *The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}. \tag{2}$$

for any real number x.

For a regular uncertainty distribution $\Phi(x)$, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ, if and only if $\{\xi \leq \Phi^{-1}(\alpha)\} = \alpha$ for all $\alpha \in [0, 1]$.

Definition 3. [28] *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let T be a totally ordered set (e.g. time). An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t.*

Definition 4. [29] *An uncertain process X_t is said to have an uncertainty distribution $\Phi_t(x)$ if at each time t, the uncertain variable X_t has the uncertainty distribution $\Phi_t(x)$.*

Theorem 1. [29] *(Sufficient and Necessary Condition) A function $\Phi_t^{-1}(\alpha) : T \times (0, 1) \longrightarrow \mathcal{R}$ is an inverse uncertainty distribution of independent increment process if and only if (i) at each time t, $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function; and (ii) for any times $t_2 < t_1$, $\Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha)$ is a monotone increasing function with respect to α.*

III. Methodology

A. Uncertain Accelerated Degradation Model

As mentioned previously, the stochastic process models which are suitable for the situations with a large amount of samples are not appropriate for the ADT with only few test items. In addition, from the physical point of view, the degradation can be treated as the accumulation of a large number of small external effects. Therefore, as one of the widely used stochastic processes, the wiener process is commonly applied to model the degradation process [5], [6], [32]. However, in the wiener process, almost all sample paths are continuous but non-Lipchitz functions [33], which are not appropriate for practical applications. Since almost all sample paths of Liu process from the uncertainty theory are Lipchitz continuous functions [33], it can be a better option for modeling the degradation process in these practical applications, especially under the circumstance of small samples.

To present the degradation process under small samples, the uncertain differential equation is introduced to describe the cumulative damage as [28]

$$dX(t) = f\left(t, X(t)\right) dt + g\left(t, X(t)\right) dC(t), \quad (3)$$

where, f and g are two functions, $C(t)$ is a canonical Liu process with stationary independent increments satisfying normal uncertainty distribution ($\mathcal{N}_u$) as $C(t) \sim \mathcal{N}_u(0, t)$ [31].

Without loss of generality, we consider a simple uncertain process $X(t)$ named arithmetic Liu process through setting $f(t, X(t)) = e$ and $g(t, X(t)) = \sigma$. Then, the solution of equation (3) with zero initial value is

$$X(t) = e \cdot t + \sigma C(t), \quad (4)$$

where e and σ are the drift and diffusion respectively. $X(t)$ follows the normal uncertainty distribution with mean et and variance $\sigma^2 t^2$. We further consider nonlinear cases through the time-scale transformation of $\tau(t)$, which is a monotonous increasing function of time t [6], [32]. Then the uncertain process $X(t)$ in equation (4) can be transformed into,

$$X(t) = e(s) \cdot \tau(t) + \sigma C(\tau(t)), \quad (5)$$

where $C(\tau(t))$ is an uncertain process satisfying normal uncertainty distribution as $C(\tau(t)) \sim \mathcal{N}_u(0, \tau(t))$. Hence, $X(t)$ is an uncertain variable at each time t with the normal uncertainty distribution

$$\Phi_t\left(x\right) = \left(1 + \exp\left(\frac{\pi\left(e(s) \cdot \tau(t) - x\right)}{\sqrt{3}\sigma\tau(t)}\right)\right)^{-1}. \quad (6)$$

$e(s)$ in equation (5) is also known as the degradation rate, which is related to the stress levels. The relationship between the normalized accelerated stress s and the degradation rate $e(s)$ is called the acceleration model [34], i.e.

$$e(s) = \exp\left(\alpha_0 + \alpha_1 s\right), \quad (7)$$

where, α_0 and α_1 are constant unknown parameters. Without the loss of generality, the i^{th} normalized accelerated stress level s_i is expressed as follows [35], [36]

$$s_i = \begin{cases} \dfrac{1/s'_0 - 1/s'_i}{1/s'_0 - 1/s'_H} & \text{Arrhenius model}, \\ \dfrac{\ln s'_i - \ln s'_0}{\ln s'_H - \ln s'_0} & \text{Power model}, \\ \dfrac{s'_i - s'_0}{s'_H - s'_0} & \text{Exponential model}, \end{cases} \quad (8)$$

where, s'_i is the i^{th} accelerated stress level, s'_0 and s'_H are the normal and highest stress levels. For simplicity, the proposed uncertain accelerated degradation model in equations (5), (7), and (8) is denoted by UADM.

Different choice of acceleration models can be made according to the type of the accelerated stress. For example, if the accelerated stress is temperature, the Arrhenius model is chosen as the acceleration model; if the accelerated stress is electrical stress, the Power model is chosen as the acceleration model [1], [4], [5].

B. Reliability and lifetime distribution of the proposed UADM

Let ω be the failure threshold for the degradation process. Generally, the lifetime of T the degradation process $X(t)$ is defined as the first time when $X(t)$ exceeds the pre-given failure threshold ω, i.e. the first hitting time (FHT). Its uncertainty distribution [30] is expressed as follows,

$$\Upsilon\left(z\right) = \mathcal{M}\left\{t_\omega \le z\right\} = \mathcal{M}\left\{\sup_{0\le t\le z} X(t) \ge \omega\right\}, \quad (9)$$

where, $\mathcal{M}\{\cdot\}$ is the uncertain measure. In addition, the time-scale transformation $\tau(t)$ is assumed as $\tau(t) = t^\beta, \beta > 0$, which is a monotonous increasing function of time t [6]. If $\beta = 1$, then $X(t)$ is a linear degradation process, if not, then $X(t)$ is a nonlinear degradation process.

To derive the analytic results of equation (9), we first need to prove that $X(t)$ is an independent increment uncertain process.

Proof

1) Following **Theorem 1**, the inverse uncertainty distribution of $X(t)$ can be computed as

$$\Phi_t^{-1}\left(\alpha\right) = e(s) \cdot t^{\,\beta} + \frac{\sigma t^{\,\beta}\sqrt{3}}{\pi} \ln\frac{\alpha}{1-\alpha}, \quad (10)$$

which is a continuous and strictly increasing function with respect to α at each time t.

2) For any times $0 < t_2 < t_1, \Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha)$ is

$$\begin{aligned} &\Phi_{t_1}^{-1}\left(\alpha\right) - \Phi_{t_2}^{-1}\left(\alpha\right) = \\ &e(s)({t_1}^{\beta} - {t_2}^{\beta}) + \frac{\sigma\sqrt{3}}{\pi}({t_1}^{\beta} - {t_2}^{\beta}) \ln\frac{\alpha}{1-\alpha}. \end{aligned} \quad (11)$$

Assume that $F(\alpha) = \Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha)$, then its derivative $F'(\alpha)$ is

$$F'\left(\alpha\right) = \frac{\sigma\sqrt{3}}{\pi}({t_1}^{\,\beta} - {t_2}^{\,\beta})\frac{1}{\alpha(1-\alpha)}. \quad (12)$$

Since that $\tau(t) = t^{\,\beta}$ is a monotonous increasing function of time t, we can get $({t_1}^{\,\beta} - {t_2}^{\,\beta}) > 0$. Therefore, it is easy to prove that $F'(\alpha) > 0$ for any $\alpha \in (0, 1)$, in other word, $\Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha)$ is a monotone increasing function with respect to α. According to **Theorem 1**, $X(t)$ is an independent increment uncertain process.

Following the extreme value theorem [30], equation (9) is referred to ,

$$\Upsilon\left(z\right) = 1 - \inf_{0\le t\le z}\left(1 + \exp\left(\frac{\pi\left(e(s) \cdot t^\beta - \omega\right)}{\sqrt{3}\sigma t^\beta}\right)\right)^{-1}. \quad (13)$$

It is known that $\tau(t) = t^{\,\beta}$ is a monotonous increasing function of time t. Thus, equation (13) is

$$\Upsilon\left(z\right) = \left(1 + \exp\left(\frac{\pi\left(\omega - e(s) \cdot z^\beta\right)}{\sqrt{3}\sigma z^\beta}\right)\right)^{-1}. \quad (14)$$

Zeng et al. [23] defined a new reliability index based on the uncertainty theory, i.e. belief reliability R_B. R_B represents the belief degree (not frequency) that a component or system will perform a required function at the specific time t under stated operating conditions. In this paper, the uncertainty distribution of R_B can be expressed as follows,

$$\begin{aligned} R_B(t) &= 1 - \mathcal{M}\{t_\omega \leq t\} \\ &= 1 - \Upsilon(t) \\ &= \left(1 + \exp\left(\frac{\pi\left(e(s)\cdot t^\beta - \omega\right)}{\sqrt{3}\sigma t^\beta}\right)\right)^{-1}. \end{aligned} \tag{15}$$

Meanwhile, the Belief Reliable Life $BL(\alpha)$ [23] is defined as the supremum lifetime given that $R_B(t)$ is larger than the belief value $\alpha \in [0,1]$, i.e.

$$BL(\alpha) = \sup\{t | R_B(z) \geq \alpha\}. \tag{16}$$

Remark 1: R_B presents the reliability evaluations under small samples with the belief degree through uncertainty theory, but not under large samples with the probability through probability theory.

Remark 2: According to the maximum uncertainty principle [22], $BL(0.5)$ is regarded as the lifetime with the maximum uncertainty.

Remark 3: Belief degree represents the strength with which we believe the event will happen. If we completely believe the event will happen, then the belief degree is 1 (complete belief). If we think it is completely impossible, then the belief degree is 0 (complete disbelief). The higher the belief degree is, the more strongly we believe the event will happen [22].

C. Uncertain Statistic for Parameter Estimations

Based on the loading profiles, there are several different types of ADT plans, including the constant stress accelerated degradation testing (CSADT), the step-stress accelerated degradation testing (SSADT), and the progressive stress accelerated degradation testing (PSADT). Since CSADT is the fundamental kinds of ADT, and its parameter estimation methods and reliability evaluation results can be extended to SSADT or PSADT, we only build up the methodology for CSADT scenario in this paper. Let x_{ijk} be the k^{th} degradation value of unit j under the i^{th} stress level and t_{ijk} be the corresponding measurement time, $i = 1, 2, \ldots, K; j = 1, 2, \ldots, n_i; k = 1, 2, \ldots, m_{ij}$, where K is the number of stress levels, n_i is the number of test samples under the i^{th} stress level and m_{ij} is the number of measurements for unit j under the i^{th} stress level.

In [22], Liu proposed the uncertain statistics method to determine the uncertainty distribution of uncertain variables with belief degrees, which is named as the principle of least squares. It can estimate unknown parameters by minimizing the sum of the squares of the distance between the obtained belief degrees and the assumed uncertainty distribution. Since the uncertain process is a sequence of uncertain variables indexed by time, its unknown parameters can also be estimated by the principle of least squares. In addition, different from the belief degrees obtained by subjective measure in [22], the belief degrees are obtained by objective measures in this paper based on the objectively observed degradation data.

Given from equation (6), the degradation variable $x_{ik} = (x_{i1k}, x_{i2k}, \ldots, x_{ijk}, \ldots)$ under the i^{th} stress level at the k^{th} monitor time is an uncertain variable following the normal uncertainty distribution. Each clement of the uncertain variable x_{ik} has a belief degree α_{ijk}. The procedure to estimate unknown parameters of the proposed UADM is as follows:

Step 1: construct belief degrees for the uncertain variables x_{ik} at the k^{th} monitor time under the i^{th} stress level based on the observed ADT data.

1) Sort the elements of $x_{ik} = (x_{i1k}, x_{i2k}, \ldots, x_{ijk}, \ldots)$ in ascending order. Define N_{ik} be the number of elements in x_{ik} (the upper bound is n_i).
2) Record the belief degrees α_{ijk} of the uncertain variable x_{ik}. In classical mathematical statistics, empirical distribution function method is used as a methodology for constructing distribution for data. When the data are scarce due to the small samples, some modification equations can be treated as suggestions for providing belief degrees on each datum. For instance, the approximate median rank can assign them with equally intervals, i.e. $\alpha_{ijk} = (j - 0.3)/(N_{ik} + 0.4), j = 1, 2, \ldots, N_{ik}$. If only one sample is tested at each stress level during ADT, i.e. $n_i = N_{ik} = 1$, then $\alpha_{ijk} = 0.5$, which is in accordance with the maximum uncertainty principle.

Step 2: Compute the objective function Q of the distance between the obtained belief degrees and the assumed uncertainty distribution from equation (6).

$$Q = \min_\theta \sum_{i=1}^{K} \sum_{k=1}^{m_{ij}} \sum_{j=1}^{N_{ik}} \left(\Phi\left(x_{ijk}\right) - \alpha_{ijk}\right)^2 \tag{17}$$

Step 3: Estimate the unknown parameter set $\theta = (\alpha_0, \alpha_1, \sigma, \beta)$ through minimizing Q.

IV. Case Study

In this section, the stress relaxation ADT data for electrical connector due to the excessive stress loss [36], [37] and a simulation case are both used to illustrate the proposed methodology. Discussions are given to explore the sensitivity of the proposed methodology to the sample sizes and verify its validity under different sample sizes.

A. The Stress Relaxation ADT Case

1) CSADT Settings : The stress relaxation case is first introduced by Yang [37], and the ADT data are provided by Ye et al [36]. Details about the case are given in Table I and the observed data are shown in Fig.1.

TABLE I
BASIC INFORMATION OF THE STRESS RELAXATION CASE

Content	Values
Accelerated stress levels (Temperature/ °C)	65, 85, 100
Normal stress level (°C)	40
Sample size under each stress level	6, 6, 6
Failure threshold (%)	30

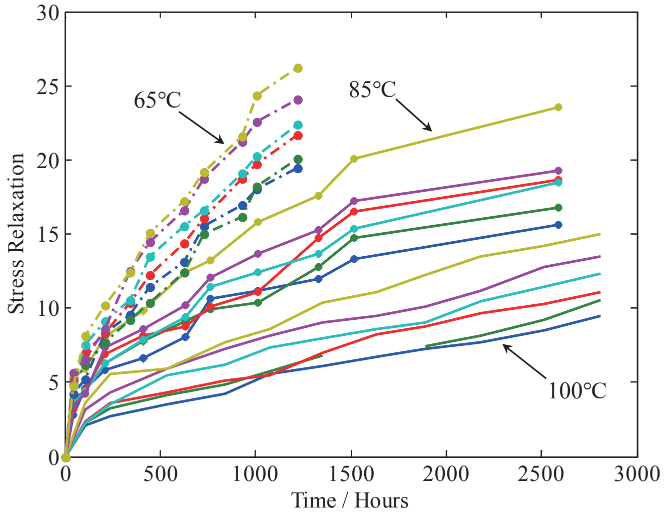

Fig. 1. The degradation paths of the stress relaxation case

2) Reliability and Lifetime Evaluations : It can be noticed that there are only six test connectors under each accelerated stress level in the stress relaxation case; therefore, it is a typical small sample problem and the ADT data contains the epistemic uncertainty. So the methodology proposed in Section III is suitable for the ADT modeling and reliability and lifetime evaluations of this case.

Given from Fig.1, the degradation processes experience nonlinear patterns. Hence, a time-scale transformation as $\tau(t) = t^{\beta}$ is used [36], [38], which is a monotonous increasing function of time t. In addition, the Arrhenius model is chosen as the acceleration model in equation (8), because the accelerated stress is temperature [34].

Unknown parameters of the proposed UADM in Section III-A are estimated through the principle of least squares in Section III-C with the stress relaxation ADT data. Results are listed in Table II.

TABLE II
PARAMETER ESTIMATIONS OF THE UADM (STRESS RELAXATION CASE)

Parameters	α_0	α_1	σ	β	Q
Values	−2.0251	1.8626	0.1195	0.4496	2.8342

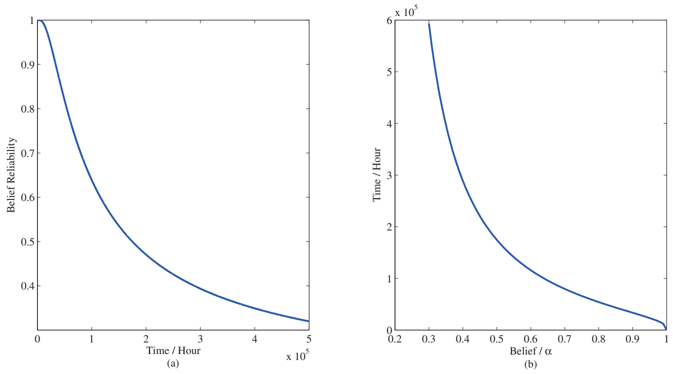

Fig. 2. Belief reliability and belief reliable life evaluations (stress relaxation case)

Then, substituting the estimated parameters into to equations (15) and (16), the belief reliability R_B and belief reliable life $BL(\alpha)$ at the use conditions can be evaluated, as shown in Fig.2 (a) and (b). Given from Fig.2 (a), if R_B = 0.9 is that of interest for decision-making, the corresponding belief reliable life $BL(0.9)$ = 33,633 (Hours), which means that products are expected to survived at the use conditions after 33,633 hours with the belief of 0.9. If R_B = 0.5, then the corresponding belief reliable life $BL(0.5)$ = 174,646 (Hours), which means that it has the maximum uncertainty that products could be survived after 174,646 hours at the use conditions.

3) Discussions : In engineering applications, decision makers prefer to choose more stable and less sensitive reliability evaluation results. In this section, the stress relaxation case is applied to explore the sensitivity of the proposed methodology to the sample sizes.

Models used for comparisons

For comparison, six different models are applied to explore their sensitivities to the sample sizes, including the UADM proposed in this paper, the Wiener process based accelerated degradation model (WADM) [6], and the Bayesian-Wiener process based accelerated degradation model (B-WADM) [38] with 4 different choices for parameters' priors.

The WADM is built up on the probability theory. In this model, the degradation process can be modeled as follows:

$$X(t) = e(s) \cdot \tau(t) + \sigma B(\tau(t)), \tau(t) = t^{\beta}, \beta > 0 \qquad (18)$$

where e and σ are called the drift and diffusion parameters. $B(\tau(t))$ is a nonlinear stochastic process satisfying the normal probability distribution ($\mathcal{N}_p$) as $B(\tau(t)) \sim \mathcal{N}_p(0, \ \tau(t))$. $\tau(t)$ is the same as equation (5), and $e(s)$ is the same as equation (7).

Based on the WADM, the B-WADM furtherly utilized the Bayesian method to present the epistemic uncertainty by assigning priors to the unknown parameters. Details about these models are shown in Table III.

TABLE III
MODELS FOR COMPARISONS (STRESS RELAXATION CASE)

Model	**Parameter estimations**	**Priors**	
		Mean	**Variance**
UADM	Principle of least squares	N/A	N/A
WADM	Maximum likelihood estimation method	N/A	N/A
B-WADM 1	Bayesian method$^{(a)}$	$100\%\mu_p^{(b)}$	$\sigma_p^{2\,(c)}$
B-WADM 2		$50\%\mu_p$	σ_p^2
B-WADM 3		$200\%\mu_p$	σ_p^2
B-WADM 4		$150\%\mu_p$	σ_p^2

Remark (a): In Bayesian method, the priors of α_0, α_1, and β are assumed to follow normal probability distributions, and the prior of σ is assumed to follow a gamma probability distribution.

Remark (b): μ_p refers to the maximum likelihood estimates of unknown parameter (See Table I in [38]). In addition, since the unknown parameter β represents the power exponent of the time-varying transformation, it will not have significant difference in different models. Thus, it is not appropriate to change the mean of the priors of β in B-WADM 1 $\sim$ 4 in Table III by percent just like other unknown parameters.

Remark (*c*): According to [38], in the stress relaxation case, the variance of the priors, i.e. σ_p^2, are set to be 0.01.

To simulate the situations under different sample sizes (from 2 to 6), we labelled the samples at each stress level in Fig.1 from 1 to 6, and then randomly select n (n=2, 3, 4, 5, 6) samples. Therefore, under each sample size (from 2 to 6), there are C_6^n combinations of samples. For instance, if n = 2, there are C_6^2=15 combinations of samples, which are 1&2, 1&3, …,3&4, …, 5&6. "1&2" means that the ADT data of samples labeled 1 and 2 under each stress level are chosen for the reliability evaluations using the models in Table III.

Parameter estimations

Before applying these models to evaluate the reliability, we need to estimate the unknown parameters first. For each model, since there are a total of ($C_6^2 + C_6^3 + C_6^4 + C_6^5 + C_6^6$=47) combinations of samples, we can get a total of 47 different groups of parameter estimation results correspondingly. For simplicity and without of generality, we calculate the average of the parameter estimation results for the UADM and the WADM, and the average of the posterior means for the B-WADM $1 \sim 4$. Then we take these averages as the final parameter estimation results for comparison. Results are shown as follows in Table IV,

TABLE IV
FINAL PARAMETER ESTIMATION RESULTS (STRESS RELAXATION CASE)

Models	Parameters	α_0	α_1	β	σ
UADM	Averages of Estimations	-2.03	1.86	0.46	0.1
WADM	Averages of Estimations	-2.28	2.01	0.46	0.4
B-WADM 1	Prior Means	-2.28	2	0.47	0.4
	Averages of Posterior Means	-2.31	1.98	0.48	0.4
B-WADM 2	Prior Means	-1.14	1	0.4	0.2
	Averages of Posterior Means	-1.20	1.05	0.44	0.5
B-WADM 3	Prior Means	-4.56	4	0.6	0.9
	Averages of Posterior Means	-4.46	4.05	0.49	0.5
B-WADM 4	Prior Means	-3.42	3	0.55	0.68
	Averages of Posterior Means	-3.39	2.98	0.49	0.48

From the results of means of priors and posterior means for B-WADM $1 \sim 4$ in Table IV, it can be recognized that the corresponding posterior estimations results are around the mean of the priors. In the Bayesian theory, the information used for estimating unknown parameters originates from two parts: prior information and testing information. Due to the small sample problem, the testing information provided by the ADT data are quite limited, which leads to the situation that the posteriors of the unknown parameters highly depend on the prior information. Unfortunately, these priors are usually directly predetermined by engineers in real applications.

In this case, the MLE in Table I in [38] are calculated based on the testing information of the stress relaxation case. In B-WADM $2 \sim 4$, the prior information are far away from MLE, so the posteriors are almost completely determined by the priors while have no connection with the testing information, which causes the ADT data to contribute little in the parameter estimations and reliability evaluations. Since ADT modeling focuses on using the ADT data to get more reliable reliability evaluation results, we will only make comparisons on the reliability evaluation results obtained by UADM, WADM, and B-WADM 1 in the following sections.

Reliability evaluations

For each model (UADM, WADM, or B-WADM 1) under the sample size n (n=2 to 6), there are C_6^n different parameter estimations that will cause C_6^n different reliability evaluation results correspondingly, so there should be the lower and upper boundaries of reliability evaluation results (when n=6, C_6^6=1, the lower and upper boundaries are the same one), which can be used to present the variation range of reliability evaluations. The lower and the upper boundaries of reliability evaluation results can be calculated as follows:

$$\begin{array}{l} \text{Lower boundary}: \; R_n^L(t) = \min\{R_n^l(t)\}, \\ \text{Upper boundary}: \; R_n^L(t) = \max\{R_n^l(t)\}. \end{array} \tag{19}$$

where $l = 1, 2, \ldots, C_6^n$.

At each monitoring time t under the sample size n, $R_n^L(t)$ represents the minimum reliability evaluations, $R_n^U(t)$ represents the maximum reliability evaluations, and $R_n^l(t)$ represents the l^{th} reliability evaluations. The results are illustrated in Fig.3.

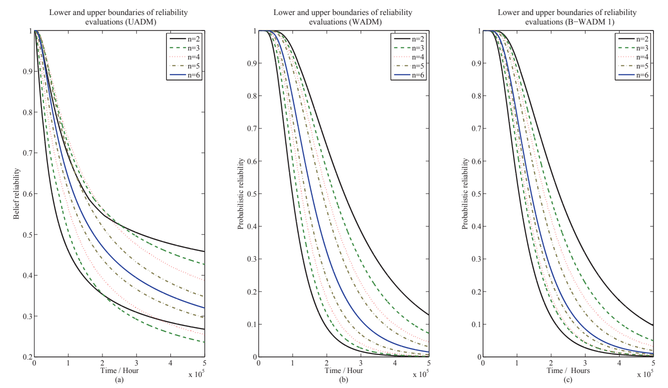

Fig. 3. Lower and upper boundaries of reliability evaluations (stress relaxation case)

Given from Fig.3, it can be seen that the belief reliability evaluations obtained by the UADM, the probabilistic reliability evaluations obtained by the WADM and the B-WADM 1 are all changing from the initial value which equals to 1; and then decrease gradually with the increasing time. For each model, the distance between the lower and upper reliability boundaries decrease with the increasing sample size from 2 to 5, which indicates that providing more information will significantly decrease the epistemic uncertainty of the reliability evaluation results. In addition, all reliability curves cover the one of n = 6. All these analyses agree with the intuitive cognition of people.

To depict the sensitivities of these models to the sample sizes more clearly, we calculate the quantile reliable lifetime for the lower and the upper reliability boundaries for each model under sample size n, and define an evaluation criteria named "the Range of quantile Reliable Lifetime under specific reliability (RRL)". The meaning of this criteria is the difference between the corresponding time $t_n^U(R)$ on the upper boundary curve and the corresponding time $t_n^L(R)$ on the lower boundary curve at a specific reliability R under the sample size n. RRL can be calculated by the following equation:

$$\begin{aligned} &RRL_n(R) = t_n^U(R) - t_n^L(R),\ n = 2,3,4,5; \\ &if\ n = 6,\ RRL_n(R) = 0. \end{aligned} \tag{20}$$

Remark: Under the sample size n, the smaller the $RRLs$ are, the smaller the variation range of the reliability evaluation is, then, the more stable the reliability evaluation results are.

For instance, at a specific reliability R under the sample size n=2, the RRL can be calculated as follows in Fig.4.

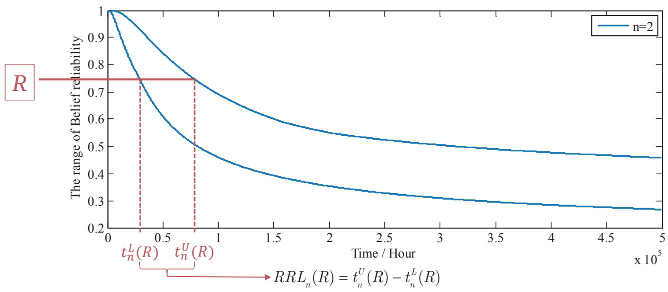

Fig. 4. An example for calculating RRL

In practical applications, the reliability from 0.8 to 1 makes more significance. So we calculate the $RRLs$ of the three models (UADM, WADM and B-WADM 1) from R=0.8 to R=0.99 (the interval is 0.001, and the total amount of $RRLs$ for each model under each sample size is 191). The results are shown in Fig.5,

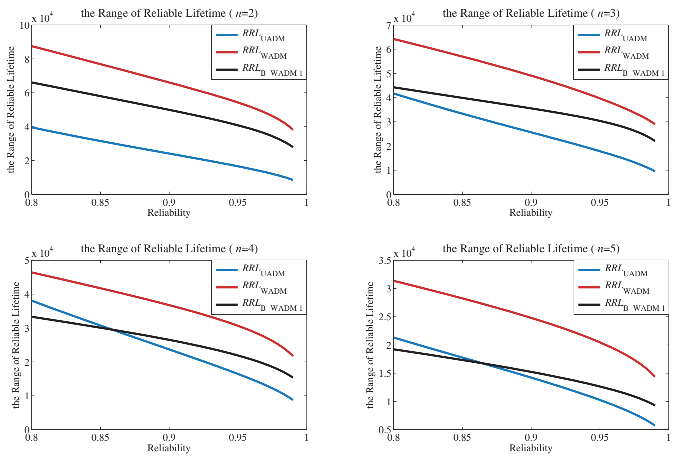

Fig. 5. the $RRLs$ under sample sizes $n(n = 2,3,4,5)$ (stress relaxation case)

From the results of Fig.5, the followings could be true:

1) When we compare both UADM and B-WADM 1 with WADM, it can be seen that both the $RRLs$ obtained by the UADM and the B-WADM 1 are obviously smaller than the ones obtained by the WADM under small sample size (from 2 to 5), which indicates that the taking the epistemic uncertainty into account can get more stable and less sensitive reliability evaluation results.
2) Furtherly, the results of Fig.5 also present that in most cases, the $RRLs$ obtained by the proposed UADM are smaller than the ones obtained by the B-WADM 1 under small sample sizes (from 2 to 5). More specifically, the $RRLs$ obtained by the proposed UADM are obviously smaller than the $RRLs$ obtained by the B-WADM 1 under extreme small sample size (from 2 to 3). When the sample size increase to 4 and 5, it can be seen from Fig.5 that under high reliability ($R > 0.87$), the $RRLs$ obtained by the proposed UADM are still smaller than the $RRLs$ obtained by the B-WADM 1.

B. The Simulation Case

For further researches of the sensitivity of the proposed methodology to sample sizes, in this section, a numerical simulation case is conducted under a wider range of sample sizes (from 3 to 20). For comparison, the WADM and B-WADM 1 $\sim$ 4 mentioned in Section IV-A3 are also used in this case.

1) Simulation Settings : Considering that all the information of the simulated degradation process are pre-given in the simulation case, there is no epistemic uncertainty when generating simulation ADT data. Therefore, the degradation model in equation (18) is chosen for generating simulation ADT data. Details about this case are listed as follows in Table V.

TABLE V
BASIC INFORMATION OF THE SIMULATION CASE

Content	**Values**
Accelerated stress	Temperature
Stress levels (°C)	$s_0' = 25, s_1' = 50, s_2' = 65, s_3' = 80$
Failure Threshold	40
Inspection interval (hours)	1000
Number of measurement	33, 23, 11
Degradation model	$X(t) = e(s)\cdot t^{\beta} + \sigma B(t^{\beta})$,
Acceleration model	$e(s) = \exp(\alpha_0 + \alpha_1 s)$
Model parameters	$\alpha_0 = -5.441, \alpha_1 = 2.716, \sigma = 0.03,$ $\beta = 0.5$
Sample size n	3, 4, 5, ..., 20
Number of Simulation l	500

2) Discussions : In this section, all the models used in Section IV-A3 are applied for further researches of their sensitivity on the sample sizes, including UADM, WADM, and B-WADM 1 $\sim$ 4.

Models used for comparison

Details about these models are shown in Table VI.

TABLE VI
MODELS FOR COMPARISONS (STRESS RELAXATION CASE)

Model	Parameter estimations	Priors	
		Mean	Variance
UADM	Principle of least squares	N/A	N/A
WADM	Maximum likelihood estimation method	N/A	N/A
B-WADM 1	Bayesian method$^{(d)}$	$100\%\mu_p^{(e)}$	$\sigma_p^{2(f)}$
B-WADM 2		$50\%\mu_p$	σ_p^2
B-WADM 3		$200\%\mu_p$	σ_p^2
B-WADM 4		$150\%\mu_p$	σ_p^2

Remark (d): In Bayesian method, the priors of α_0, α_1, and β are assumed to follow normal probability distributions, and the prior of σ is assumed to follow a gamma probability distribution.

Remark (e): μ_p refers to the setting values of model parameters(See Table V). Since the unknown parameter β represents the power exponent of the time-varying transformation, it will not have significant difference in different models. Thus, it is not appropriate to change the mean of the priors of β in B-WADM $1 \sim 4$ in Table III by percent just like other unknown parameters.

Remark (f): or simplicity and calculation, in the simulation case, the variance of the priors, i.e. σ_p^2, are set to be 0.01.

Parameter estimations

First of all, the unknown parameters of these models need to be estimated. As shown in Table V, we can get a total of 500 different parameter estimation results for each model u each sample size from 3 to 20. For simplicity and witho generality, we take the average of the parameter estim results for the UADM and the WADM, and the average c posterior means for the B-WADM $1 \sim 4$ as the final parai estimation results. Results are shown as follows in Table

TABLE VII
FINAL PARAMETER ESTIMATION RESULTS (STRESS RELAXATION C.

Models	Parameters	α_0	α_1	β	
UADM	Averages of Estimations	-5.43	2.71	0.50	0.
WADM	Averages of Estimations	-5.45	2.72	0.5	0.030
B-WADM 1	Prior Means	-5.441	2.72	0.5	0.03
	Averages of Posterior Means	-5.43	2.72	0.50	0.030
B-WADM 2	Prior Means	-2.72	1.36	0.45	0.015
	Averages of Posterior Means	-4.23	1.98	0.44	0.046
B-WADM 3	Prior Means	-10.88	5.43	0.60	0.06
	Averages of Posterior Means	-10.54	5.62	0.46	0.22
B-WADM 4	Prior Means	-8.16	4.07	0.55	0.045
	Averages of Posterior Means	-7.53	4.13	0.58	0.026

Similar to the results of Section 4.1.3.2 in Table IV, the average of posteriors means estimated by B-WADM $2 \sim 4$ are far away from the testing information (the setting values of model parameters), which means that the ADT data have little effect on the parameter estimations and reliability evaluations. So in the following sections, only the reliability evaluation results obtained by UADM, WADM, and B-WADM 1 are used for comparison.

Reliability evaluations

According to equation (19) $(l = 1, 2, \ldots, 500)$, the lower and upper boundaries of reliability evaluation results under sample size $n(n = 3, 4, \ldots, 20)$ are calculated and used to present the variation range of reliability evaluation results for each model. Results are illustrated in Fig.6 and Fig.7.

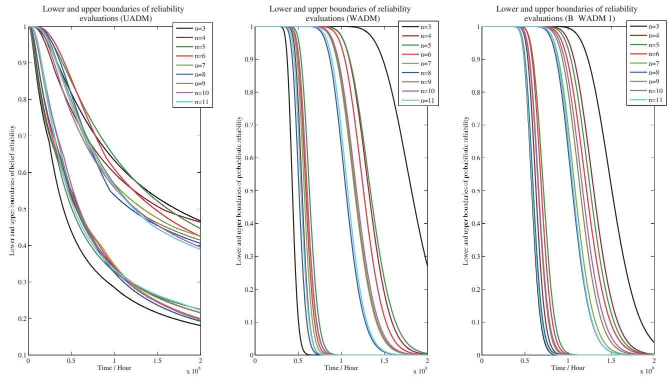

Fig. 6. Lower and upper boundaries of reliability evaluations (simulation case, $n = 3 \sim 11$)

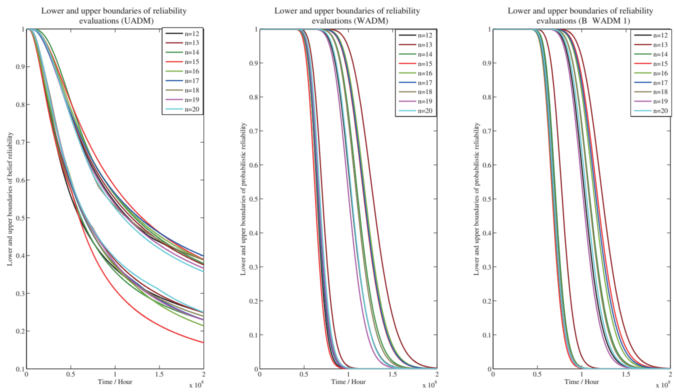

Fig. 7. Lower and upper boundaries of reliability evaluations (simulation case, $n = 12 \sim 20$)

According to the results of Fig.6 and Fig.7, under sample size n (from 3 to 20), the reliability evaluations obtained by UADM, WADM, and B-WADM 1 all starts from the initial value 1 and then decrease with the increasing time t. For each model, with the increasing sample size, the distance between the lower and upper reliability boundaries decreases gradually, which indicates that the epistemic uncertainty decreases when more information are provided. All these analyses agree with the intuitive cognition of people.

To depict the sensitivities of these models to the sample sizes more clearly, based on equation (20) $(n = 3, 4, \ldots, 20)$, we calculate the he $RRLs$ of the three models (UADM, WADM and B-WADM 1) from $R = 0.8$ to $R = 0.99$ under each sample size from 3 to 20 (the interval is 0.001, and the total amount of $RRLs$ for each model under each sample size is 191).The results are shown as follows in Fig.8.

According to the results of Fig.8., the following conclusions could be true:

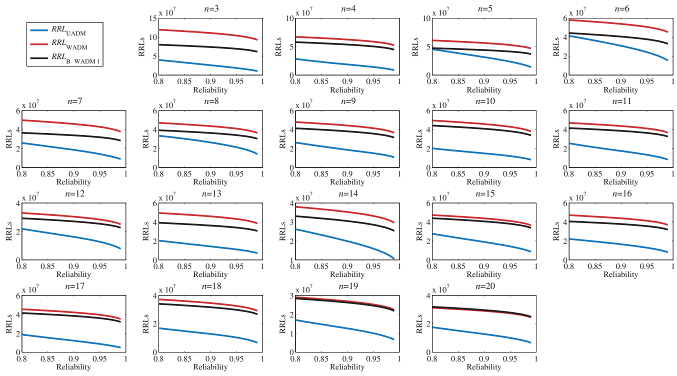

Fig. 8. the $RRLs$ under sample size $n(n = 3 \sim 20)$ (simulation case)

1) In this simulation case, under a wide range of sample size from 3 to 20, the $RRLs$ obtained by the UADM and the B-WADM 1 are both smaller than the ones obtained by the WADM. The results show that more stable and less sensitive reliability evaluations could be obtained if the epistemic uncertainty is considered.
2) Furtherly, under each sample size from 3 to 20, the $RRLs$ obtained by the proposed UADM are smaller than the ones obtained by the B-WADM 1. The results indicates that although proposed UADM and the B-WADM 1 both take the epistemic uncertainty into account, the proposed UADM can obtain more stable and less sensitive reliability evaluations than the B-WADM 1.

The analysis results of the stress relaxation case and the simulation case both show that the reliability evaluation results obtained by the UADM are more stable and less sensitive than the ones obtained by the WADM and the B-WADM under the small sample situation. Consequently, the proposed UADM will contribute a more concrete and suitable support for engineers to control the risk and make appropriate decisions, such as maintenance strategies.

V. Conclusions

This paper concentrates on dealing with the small sample problem in ADT data, and draws the following conclusions:

1) The arithmetic Liu process based on the uncertainty theory is introduced to conduct an uncertain accelerated degradation model, namely UADM, to capture the epistemic uncertainty due to small samples in ADT data. The reliability and lifetime evaluations are derived correspondingly.
2) The uncertain statistics method for the proposed uncertain accelerated degradation model is given to estimate the unknown parameters. In this method, the belief degrees are obtained by objective measures based on the objectively observed degradation data, rather than by subjective measures.
3) The results of the practical case show the practicability of the proposed methodology. The discussion results of the practical case and the simulation case show that under small sample sizes, the proposed uncertain accelerated degradation model provides more stable and less sensitive reliability evaluations than the Wiener process based accelerated degradation model (WADM) and the Bayesian-Wiener process based accelerated degradation model (B-WADM). The analysis results indicate that the proposed model is an appropriate option for the small sample situation, and will furtherly contribute a more concrete and suitable support for engineers to control the risk and make appropriate decisions, such as maintenance strategies.

Beyond the work of this paper, there are other issues that may be worthwhile for future researches: this paper is the first try to introduce the uncertainty theory to the field of ADT modeling and uses a practical case and a simulation case. In the future, it is very important to explore the appropriate data sizes (including the number of measurements and the number of test items) for the application of the proposed model by applying to more experimental cases.

Acknowledgment

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A New Method of Level-2 Uncertainty Analysis in Risk Assessment Based on Uncertainty Theory

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Abstract The objective of this study is to present a novel method of level-2 uncertainty analysis in risk assessment by means of uncertainty theory. In the proposed method, aleatory uncertainty is characterized by probability distributions, whose parameters are affected by epistemic uncertainty. These parameters are described as uncertain variables. For monotone risk models, such as fault trees or event trees, the uncertainty is propagated analytically based on the operational rules of uncertain variables. For non-monotone risk models, we propose a simulation-based method for uncertainty propagation. Three indexes, i.e., average risk, Value-at-risk and bounded value-at-risk, are defined for risk-informed decision making in the level-2 uncertainty setting. Two numerical studies and an application on a real example from literature are worked out to illustrate the developed method. A comparison is made to some commonly-used uncertainty analysis methods, e.g., the ones based on probability theory and evidence theory.

Keywords Uncertainty theory · Uncertainty analysis · Epistemic uncertainty

1 Introduction

Uncertainty modeling and analysis is an essential part of probabilistic risk assessment (PRA) and has drawn numerous attentions since 1980s (Apostolakis, 1990; Parry and Winter, 1981). Two types of uncertainty are usually distinguished: aleatory uncertainty, which refers to the uncertainty inherent in the physical behavior of a system, and epistemic uncertainty, which refers to the uncertainty in the modelling caused by lack of knowledge on the system behavior (Kiureghian and Ditlevsen, 2009). In practice, uncertainty modeling and analysis involving both aleatory and epistemic uncertainty is often formulated in a level-2 setting: aleatory uncertainty is considered by developing probabilistic models for risk assessment, while the parameters in the probabilistic models might subject to epistemic uncertainty (Aven et al, 2014).

In general, it has been well acknowledged that aleatory uncertainty should be modeled using probability theory. However, there appears to be no consensus on which mathematical framework should be used to describe epistemic uncertainty, since its modeling usually involves subjective information from human judgements. Indeed, various mathematical frameworks have been proposed in the literature to model the epistemically uncertain variables, e.g., probability theory (subjective interpretation), evidence theory, possibility theory, etc. (Aven, 2013; Aven and Zio, 2011; Helton et al, 2010). As a result, different methods for level-2 uncertainty analysis are developed. Aven et al (2014) systematically elaborate on level-2 uncertainty analysis methods and developed a purely probabilistic for level-2 uncertainty analysis. Limbourg and Rocquigny (2010) apply evidence theory to both level-1 and level-2 uncertainty modeling and analysis, and the two settings were compared through a benchmark problem. Some explanations of the results are discussed in the context of evidence theory. Considering the large calculation cost for level-2 uncertainty analysis, Limbourg et al (2010)

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develop an accelerated method for monotonous problems using the Monotonous Reliability Method (MR-M). Pedroni et al (2013) and Pedroni and Zio (2012) model the epistemic uncertainty using possibility distributions and develop a level-2 Monte Carlo simulation for uncertainty analysis, which is then compared to a purely probabilistic approach and an evidence theory-based (ETB) approach. Pasanisi et al (2012) reinterpret the level-2 purely probabilistic frameworks in the light of Bayesian decision theory and apply the approach to risk analysis. Hybrid methods based on probability theory and evidence theory are also presented (Aven et al, 2014). Baraldi et al (2013) introduce the hybrid level-2 uncertainty models to consider maintenance policy performance assessment.

In this paper, we enrich the research of level-2 uncertainty analysis by introducing a new mathematical framework, the uncertainty theory, to model the epistemically uncertain variables. Uncertainty theory has been founded in 2007 by Liu (2007) as an axiomatic mathematical framework to model subjective belief degrees. It is viewed as a reasonable and effective approach to describe epistemic uncertainty (Kang et al, 2016). To simulate the evolution of an uncertain phenomenon with time, concepts of uncertain process (Liu, 2015) and uncertain random process (Gao and Yao, 2015) are proposed. The uncertain differential equation is also developed as an effective tool to model events affected by epistemic uncertainty (Yang and Yao, 2016). After these years of development, uncertainty theory has been applied in various areas, including finance (Chen and Gao, 2013; Guo and Gao, 2017), decision making under uncertain environment (Wen et al, 2015b,a), game theory (Yang and Gao, 2013, 2016; Gao et al, 2017; Yang and Gao, 2014), etc. There are also considerable real applications in reliability analysis and risk assessment considering epistemic uncertainties. For example, Zeng et al (2013) propose a new concept of belief reliability based on uncertainty theory accounting for both aleatory and epistemic uncertainties. Wen et al (2017) develop an uncertain optimization model of spare parts inventory for equipment system, where the subjective belief degree is adopted to compensate the data deficiency. Ke and Yao (2016) apply uncertainty theory to optimize scheduled replacement time under block replacement policy considering human uncertainty. Wen and Kang (2016) model the reliability of systems with both random components and uncertain components. Wang et al (2017) develop a new structural reliability index based on uncertainty theory.

To the best of our knowledge, in this paper, it is the first time that uncertainty theory is applied to level-2 uncertainty analysis. Through comparisons to some commonly used level-2 uncertainty analysis methods, new insights are brought with respect to strength and limitations of the developed method.

The remainder of the paper is structured as follows. Section 2 recalls some basic concepts of uncertainty theory. Level-2 uncertainty analysis method is developed in Section 3, for monotone and non-monotone risk models. Numerical case studies and applications are presented in Section 4. The paper is concluded in Section 5.

2 Preliminaries

In this section, we briefly review some basic knowledge on uncertainty theory. Uncertainty theory is a new branch of axiomatic mathematics built on four axioms, i.e., Normality, Duality, Subadditivity and Product Axioms. Founded by Liu (2007) in 2007 and refined by Liu (2010) in 2010, uncertainty theory has been widely applied as a new tool for modeling subjective (especially human) uncertainties. In uncertainty theory, belief degrees of events are quantified by defining uncertain measures:

Definition 1 (Uncertain measure (Liu, 2007)) *Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following three axioms,*

Axiom 1 (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ *for the universal set Γ.*

Axiom 2 (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event $\Lambda \in \mathcal{L}$.*

Axiom 3 (Subadditivity Axiom) *For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}$$

Uncertain measures of product events are calculated following the product axiom (Liu, 2009):

Axiom 4 (Product Axiom) *Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\left\{\Lambda_k\right\}$$

where $\mathcal{L}_k$ are σ-algebras over Γ_k, and Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

In uncertainty theory, if an uncertain measure of one event can take multiple reasonable values, a value as close to 0.5 as possible is assigned to the event so as to maximize the uncertainty (maximum uncertainty principle) (Liu, 2009). Hence, the uncertain measure of an arbitrary event in the product σ-algebra $\mathcal{L}$ is calculated by

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda} \min\limits_{1\le k\le\infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda} \min\limits_{1\le k\le\infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c} \min\limits_{1\le k\le\infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c} \min\limits_{1\le k\le\infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \qquad \text{otherwise.} \end{cases} \tag{1}$$

Definition 2 (Uncertain variable (Liu, 2007)) *An uncertain variable is a function ξ from an uncertainty space $(\Lambda, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in \mathcal{B}\}$ is an event for any Borel set $\mathcal{B}$ of real numbers.*

Definition 3 (Uncertainty distribution (Liu, 2007)) *The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x.*

For example, a linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an uncertainty distribution

$$\Phi_1(x) = \begin{cases} 0, & \text{if } x < a \\ \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x > b \end{cases} \tag{2}$$

and a normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an uncertainty distribution

$$\Phi_2(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ x \in \Re \tag{3}$$

An uncertainty distribution Φ is said to be regular if it is a continuous and strictly increasing with respect to x, with $0 < \Phi(x) < 1$, and $\lim\limits_{x\to-\infty} \Phi(x) = 0$, $\lim\limits_{x\to+\infty} \Phi(x) = 1$. A regular uncertainty distribution has an inverse function, and this inverse function is defined as the inverse uncertainty distribution, denoted by $\Phi^{-1}(\alpha)$, $\alpha \in (0, 1)$. It is clear that linear uncertain variables and normal uncertain variables are regular, and their inverse uncertainty distributions are written as:

$$\Phi_1^{-1}(\alpha) = (1-\alpha)a + \alpha b, \tag{4}$$

$$\Phi_2^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{5}$$

Inverse uncertainty distributions play a central role in uncertainty theory, since the uncertainty distribution of a function of uncertain variables is calculated using the inverse uncertainty distributions:

Theorem 1 (Operational law (Liu, 2010)) *Let ξ_1, $\xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to ξ_{m+1}, $\xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \Phi_n^{-1}(1-\alpha)\right). \tag{6}$$

Definition 4 (Expected value (Liu, 2007)) *Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}dx - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}dx. \tag{7}$$

It is clear that, if ξ has an uncertainty distribution $\Phi(x)$, the expected value of ξ can be calculated by (Liu, 2015):

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\, dx - \int_{-\infty}^0 \Phi(x)dx. \tag{8}$$

For ξ with a regular uncertainty distribution, the expected value $E[\xi]$ is given by (Liu, 2015)

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \tag{9}$$

3 Level-2 Uncertainty Analysis Based on Uncertainty Theory

In this section, a new method for level-2 uncertainty analysis is presented based on uncertainty theory. Sect. 3.1 formally defines the problem of level-2 uncertainty analysis. Then, the uncertainty analysis method is introduced for monotone and non-monotone models in Sect. 3.2 and 3.3, respectively.

3.1 Problem Definition

Conceptually, uncertainty analysis of a risk model can be represented as:

$$\begin{aligned} &z = g(\mathbf{x}), \\ &p = h\left(g(\mathbf{x}), z_{th}\right), \end{aligned} \tag{10}$$

where z is the safety variable of the system of interest, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is a vector of input parameters, p is the risk indicator expressed in probabilistic terms and calculated by a distance function $h(\cdot)$ between the value of z and safety threshold z_{th}:

$$p = \Pr\{z > z_{th}\} \text{ or } p = \Pr\{z < z_{th}\}. \tag{11}$$

In practice, $g(\cdot)$ could be logical models, e.g., fault trees, event trees, Bayesian networks, etc., or physical models of failure dynamics, e.g., see Baraldi and Zio (2008) and Ripamonti et al (2013).

Uncertainty in (10) is assumed to come from the input parameters $\mathbf{x}$, i.e., model uncertainty (e.g., see Nilsen and Aven (2003)) is not considered in the present paper. Aleatory and epistemic uncertainty are considered separately. Depending on the ways the uncertainty in the model parameters is handled, level-1 and level-2 uncertainty models are distinguished.

Level-1 uncertainty models separate the input vector into $\mathbf{x} = (\mathbf{a}, \mathbf{e})$, where $\mathbf{a} = (x_1, x_2, \cdots, x_m)$ represents the parameters affected by aleatory uncertainty while $\mathbf{e} = (x_{m+1}, x_{m+2}, \cdots, x_n)$ represents the parameters that are affected by epistemic uncertainty (Limbourg and Rocquigny, 2010). In level-1 uncertainty models, probability theory is used to model the aleatory uncertainty in $\mathbf{a} = (x_1, x_2, \cdots, x_m)$ by identifying their probability density functions (PDF) $f(x_i|\theta_i)$. These PDFs are assumed to be known, i.e., the parameters in the PDFs, denoted by $\mathbf{\Theta} = (\theta_1, \theta_2, \cdots, \theta_n)$, are assumed to have precise values. In practice, however, $\mathbf{\Theta} = (\theta_1, \theta_2, \cdots, \theta_n)$, are subject to epistemic uncertainty, and the corresponding uncertainty model is called level-2 uncertainty model.

In this paper, we consider the generic model in (10) and develop a new method for level-2 uncertainty analysis, based on uncertainty theory. More specifically, it is assumed that:

(1) The aleatory uncertainty in the input parameters are described by the PDFs $f(x_i|\theta_i), i = 1, 2, \cdots, n$.

(2) $\mathbf{\Theta} = (\theta_1, \theta_2, \cdots, \theta_n)$ are modeled as independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$.

The uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$ describe the epistemic uncertainty in the parameter values of $\mathbf{\Theta} = (\theta_1, \theta_2, \cdots, \theta_n)$ and can be determined based on expert knowledge, using uncertain statistical methods such as interpolation (Liu, 2015), optimization (Hou, 2014) and the method of moments (Wang and Peng, 2014). The problem, then, becomes: given $\Phi_1, \Phi_2, \cdots, \Phi_n$, how to assess the epistemic uncertainty in the risk index of interest p. In the following sections, we first develop the uncertainty analysis method for monotone models in Sect. 3.2, where p is a monotone function of the parameters $\mathbf{\Theta}$, and, then, discuss a more general case in Sect. 3.3, where there are no requirements on the monotony of the risk model.

3.2 Monotone Risk Model

3.2.1 Uncertainty Analysis Using Operational Laws

In monotone uncertainty models, the risk index of interest can be explicitly expressed as:

$$p = h(\mathbf{\Theta}), \tag{12}$$

where $\mathbf{\Theta} = (\theta_1, \theta_2, \cdots, \theta_n)$ is the vector of the parameters in the PDFs whose values are subject to epistemic uncertainty and h is a strictly monotone function with respect to $\mathbf{\Theta}$. According to Assumption (2) in Section 3.1, the risk index of interest p is also an uncertain variable. Given regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$ for $\theta_1, \theta_2, \cdots, \theta_n$, the epistemic uncertainty in p can be represented by an uncertainty distribution $\Psi(p)$. Without loss of generality, we assume h is strictly increasing with respect to $\theta_1, \theta_2, \cdots, \theta_m$, and strictly decreasing with respect to $\theta_{m+1}, \theta_{m+2}, \cdots, \theta_n$. Then, the inverse uncertainty distribution of p can be calculated based on Theorem 1, i.e.,

$$\begin{aligned}\Psi_p^{-1}(\alpha) = h(&\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \\ &\Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)), 0 \le \alpha \le 1.\end{aligned} \tag{13}$$

The uncertainty distribution $\Psi(p)$ can be obtained from the inverse function $\Psi_p^{-1}(\alpha)$.

Two risk indexes are defined for risk-informed decision making, considering the level-2 uncertainty settings presented.

Definition 5 *Let p represent a probabilistic risk index and $\Psi(p)$ be the uncertainty distribution of p. Then*

$$\bar{p} = \int_0^{+\infty} [1 - \Psi(p)]\, dp \tag{14}$$

is defined as the average risk, and

$$VaR(\gamma) = \sup\{p|\Psi(p) \le \gamma\} \tag{15}$$

is defined as the value-at-risk.

It should be noted that the average risk can be also calculated using the inverse distribution of p:

$$\bar{p} = \int_0^1 \Psi^{-1}(\alpha) d\alpha, \tag{16}$$

and the value-at-risk can also be calculated by

$$\mathrm{VaR}(\gamma) = \Psi^{-1}(\gamma). \tag{17}$$

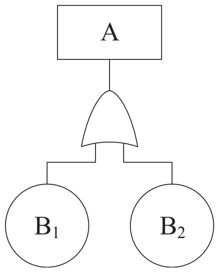

Fig. 1 Simple fault tree for the case study

According to Definition 5, the average risk is the expected value of the uncertain variable p, which reflects our average belief degree of the risk index p. A greater value of the average risk indicates that we believe the risk is more severe. The physical meaning of value-at-risk is that, with belief degree γ, we believe that the value of the risk index is p. It is clear that, for a fixed value of γ, a greater VaR(γ) means that the risk is more severe.

3.2.2 Numerical Case Study

We take a simple fault tree (shown in Fig. 1) as a numerical case study to demonstrate the application of the developed method. The fault tree represents a top event A as the union (logic gate OR) of the two basic events B_1 and B_2. The risk index of interest is the probability that event A occurs before time t_0, determined by

$$p = \Pr\{t_A < t_0\}, \tag{18}$$

where t_A denotes the occurrence time of A. Let t_{B1} and t_{B2} be the occurrence times of events B_1 and B_2, respectively. Then, $t_A = \min(t_{B1}, t_{B2})$. Assume that t_{B1} and t_{B2} follow exponential distributions with parameters λ_1 and λ_2, respectively. Thus, (5) can be further expressed as:

$$\begin{aligned} p &= \Pr\{t_{B1} < t_0, t_{B2} < t_0\} \\ &= p_{B1} + p_{B2} - p_{B1} \cdot p_{B2} \\ &= 1 - e^{-(\lambda_1+\lambda_2)t_0} \end{aligned} \tag{19}$$

It is assumed that λ_1 and λ_2 are subject to epistemic uncertainty. The developed methods in Sect. 3.2.1 are used for level-2 uncertainty analysis based on uncertainty theory. In accordance with expert experience, linear uncertainty distributions are used to model the epistemic uncertainty in λ_1 and λ_2, i.e., $\lambda_1 \sim \mathcal{L}(a_1, b_1)$ and $\lambda_2 \sim \mathcal{L}(a_2, b_2)$. From (13), the inverse uncertainty distribution of the risk index p is calculated as

$$\Psi_p^{-1}(\alpha) = 1 - \exp\left[-(1-\alpha)(a_1+a_2)t_0 - \alpha(b_1+b_2)t_0\right], \quad 0 \le \alpha \le 1 \tag{20}$$

Table 1 Time threshold and distributions for level-2 uncertain parameters

	$\lambda_1(10^{-5}/h^{-1})$	$\lambda_2(10^{-5}/h^{-1})$	t_0	γ
UTB Method	$\mathcal{L}(0.8, 1.2)$	$\mathcal{L}(0.5, 0.8)$	10^4h	0.9
PB Method	$U(0.8, 1.2)$	$U(0.5, 0.8)$		

and the uncertainty distribution of p is

$$\Psi(p) = \begin{cases} 0, & \text{if } p \le \Psi_p^{-1}(0) \\ \dfrac{-\frac{1}{t_0}\ln(1-p) - (a_1+a_2)}{(b_1+b_2)-(a_1+a_2)}, & \text{if } \Psi_p^{-1}(0) \le p \le \Psi_p^{-1}(1) \\ 1, & \text{if } p \ge \Psi_p^{-1}(0). \end{cases} \tag{21}$$

According to (16) and (17), $\overline{p}$ and VaR can be calculated by

$$\begin{aligned} \overline{p} &= \int_0^1 \Psi^{-1}(\alpha)d\alpha \\ &= \int_0^1 1d\alpha - \int_0^1 \exp\left[(a_1+a_2-b_1-b_2)t_0\alpha - (a_1+a_2)t_0\right]d\alpha \\ &= 1 - \frac{1}{(a_1+a_2-b_1-b_2)t_0}\left[e^{-(b_1+b_2)t_0} - e^{-(a_1+a_2)t_0}\right], \end{aligned} \tag{22}$$

$$\begin{aligned} \text{VaR}(\gamma) &= \Psi_p^{-1}(\gamma) \\ &= 1 - \exp\left[-(1-\gamma)(a_1+a_2)t_0 - \gamma(b_1+b_2)t_0\right]. \end{aligned} \tag{23}$$

Assuming the parameter values in Table 1, we have $\overline{p} = 0.1519$ and VaR(0.9) $= 0.1755$. The results are compared to those from a similar method based on probability theory, hereafter indicated as probability-based method (PB), whereby the belief degrees on λ_1, λ_2 and p are modeled by random variables. In this paper, we assume that λ_1 and λ_2 follow uniform distributions whose parameter values are given in Table 1. Monte Carlo (MC) sampling is used to generate samples from the probability distribution of p. Average risk and value-at-risk can, then, be calculated using the MC samples:

$$\overline{p} = \frac{1}{n}\sum_{i=1}^{n} p_i, \tag{24}$$

$$\text{VaR}(\gamma) = \sup\{p_i | p_i \le \gamma, i = 1, 2, \cdots, n\} \tag{25}$$

where $p_i, i = 1, 2, \cdots, n$ are the samples obtained by MC simulation.

Figure 2 compares the distributions of the risk indexes obtained from the two methods. Both distributions have the same supports, but the uncertainty distribution has more weights on high values of the risk

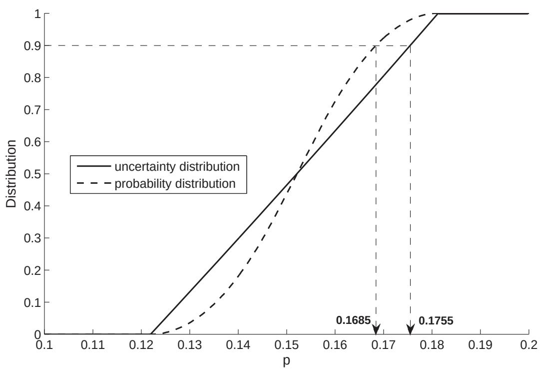

Fig. 2 Level-2 propagation results from uncertainty-theory-based (UTB, solid line) and probability-based (PB, dashed line) methods

Table 2 Risk indexes of the monotone risk model

Method	$\overline{p}$	VaR(0.9)
UTB Method	0.1519	0.1755
PB Method	0.1520	0.1685

index than the probability theory. This means that the uncertainty theory-based (UTB) method is more conservative than the PB method, since it tends to evaluate a higher risk. This is obtained also from the values in Table 2: although both methods have roughly the same $\overline{p}$, the UTB method yields a higher VaR(0.9), which indicates a high risk.

3.3 Non-monotone Risk Model

3.3.1 Uncertainty Analysis Using Uncertain Simulation

In many practical situations, the risk index of interest cannot be expressed as a strictly monotone function of the level-2 uncertain parameters. For such cases, we cannot obtain the exact uncertainty distributions for p by directly applying the operational laws. Rather, the maximum uncertainty principle needs to be used to derive the upper and lower bounds for the uncertainty distribution based on an uncertain simulation method developed by (Zhu, 2012). The uncertain simulation can provide a reasonable uncertainty distribution of a function of uncertain variables, and does not require the monotonicity of the function with respect to the variables. In this section, the method is extended to calculate the upper and lower bounds of an uncertainty distribution for risk assessment.

Definition 6 ((Zhu, 2012)) *An uncertain variable ξ is common if it is from the uncertain space $(\Re, \mathcal{B}, \mathcal{M})$ to $\Re$ defined by $\xi(\gamma) = \gamma$, where $\mathcal{B}$ is the Borel algebra over $\Re$. An uncertain vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is common if all the elements of $\boldsymbol{\xi}$ are common.*

Theorem 2 ((Zhu, 2012)) *Let $f : \Re^n \to \Re$ be a Borel function, and $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ be a common uncertain vector. Then the uncertainty distribution of f is:*

$$\Psi(x) = \mathcal{M}\left\{f(\xi_1, \xi_2, \cdots, \xi_n) \le x\right\}$$
$$= \begin{cases} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min\limits_{1 \le k \le n} \mathcal{M}_k\left\{\Lambda_k\right\}, \\ \quad if \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda} \min\limits_{1 \le k \le n} \mathcal{M}_k\left\{\Lambda_k\right\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c} \min\limits_{1 \le k \le n} \mathcal{M}_k\left\{\Lambda_k\right\}, \\ \quad if \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n \subset \Lambda^c} \min\limits_{1 \le k \le n} \mathcal{M}_k\left\{\Lambda_k\right\} > 0.5 \\ 0.5, \quad otherwise. \end{cases} \tag{26}$$

In (26), $\Lambda = f^{-1}(-\infty, x)$, $\{A_i\}$ denotes a collection of all intervals of the form $(-\infty, a]$, $[b, +\infty)$, $\varnothing$ and $\Re$, and each $\mathcal{M}_k\{\Lambda_k\}$ is derived based on (27):

$$\mathcal{M}\{B\} = \begin{cases} \inf\limits_{B \subset \bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, \\ \quad if \inf\limits_{B \subset \bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 1 - \inf\limits_{B^c \subset \bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, \\ \quad if \inf\limits_{B^c \subset \bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 0.5, \quad otherwise, \end{cases} \tag{27}$$

where $B \in \mathcal{B}$, and $B \subset \bigcup\limits_{i=1}^{\infty} A_i$.

From Theorem 2, it can be seen that (27) gives a theoretical bound of each $\mathcal{M}_k\{\Lambda_k\}$ in (26). Let $m = \inf\limits_{B \subset \bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}$, $n = \inf\limits_{B^c \subset \bigcup A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}$. It is clear that any values within m and $1 - n$ is a reasonable value for $\mathcal{M}\{B\}$. Hence, we use m as the upper bound and $1 - n$ as the lower bound of $\mathcal{M}_k\{\Lambda_k\}$ and develop a numerical algorithm for level-2 uncertainty analysis.

Algorithm 1. (Level-2 uncertainty analysis for non-monotone models)

step 1: Set $m_1(i) = 0$ and $m_2(i) = 0$, $i = 1, 2, \cdots, n$.

step 2: Randomly generate $u_k = \left(\gamma_k^{(1)}, \gamma_k^{(2)}, \cdots, \gamma_k^{(n)}\right)$ with $0 < \Phi_i\left(\gamma_k^{(i)}\right) < 1$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots, N$.

step 3: From $k = 1$ to $k = N$, if $f(u_k) \le c$, $m_1(i) = m_1(i) + 1$, denote $x_{m_1(i)}^{(i)} = \gamma_k^{(i)}$;
otherwise, $m_2(i) = m_2(i) + 1$, denote $y_{m_2(i)}^{(i)} = \gamma_k^{(i)}$, $i = 1, 2, \cdots, n$.

step 4: Rank $x^{(i)}_{m_1}$ and $y^{(i)}_{m_2}$ from small to large, respectively.

step 5: Set

$$a^{(i)} = \Phi\left(x^{(i)}_{m_1(i)}\right) \wedge \left(1-\Phi\left(x^{(i)}_1\right)\right) \wedge \left(\Phi\left(x^{(i)}_1\right)+1-\Phi\left(x^{(i)}_2\right)\right) \wedge \left(\Phi\left(x^{(i)}_{m_1(i)-1}\right)+1-\Phi\left(x^{(i)}_{m_1(i)}\right)\right);$$

$$b^{(i)} = \Phi\left(y^{(i)}_{m_2(i)}\right) \wedge \left(1-\Phi\left(y^{(i)}_1\right)\right) \wedge \left(\Phi\left(y^{(i)}_1\right)+1-\Phi\left(y^{(i)}_2\right)\right) \wedge \left(\Phi\left(y^{(i)}_{m_2(i)-1}\right)+1-\Phi\left(y^{(i)}_{m_2(i)}\right)\right).$$

step 6: $L^{(i)}_{1U} = a^{(i)}, L^{(i)}_{1L} = 1-b^{(i)}, L^{(i)}_{2U} = b^{(i)}, L^{(i)}_{2L} = 1-a^{(i)}$.

step 7: If $a_U = L^{(1)}_{1U} \wedge L^{(2)}_{1U} \wedge \cdots \wedge L^{(n)}_{1U} > 0.5$, $L_U = a_U$; if $b_U = L^{(1)}_{2L} \wedge L^{(2)}_{2L} \wedge \cdots \wedge L^{(n)}_{2L} > 0.5$, $L_U = 1-b_U$; otherwise, $L_U = 0.5$.
If $a_L = L^{(1)}_{1L} \wedge L^{(2)}_{1L} \wedge \cdots \wedge L^{(n)}_{1L} > 0.5$, $L_L = a_L$; if $b_L = L^{(1)}_{2U} \wedge L^{(2)}_{2U} \wedge \cdots \wedge L^{(n)}_{2U} > 0.5$, $L_L = 1-b_L$; otherwise, $L_L = 0.5$.

Through this algorithm, the upper and lower bounds for the uncertainty distribution of p can be constructed, denoted by $[\Psi_L(p), \Psi_U(p)]$. Similar to the monotone case, we define two risk indexes considering the level-2 uncertainty:

Definition 7 *Let p described by (11) be the probability that a hazardous event will happen, and let $\Psi_L(p)$ and $\Psi_U(p)$ be the lower bound and upper bound of the uncertainty distribution of p, respectively. Then*

$$\overline{p} = \int_0^{+\infty} \left[1 - \frac{\Psi_L(p)+\Psi_U(p)}{2}\right] dp \tag{28}$$

is defined as average risk, and

$$[VaR_L, VaR_U](\gamma) = [\sup\{p|\Psi_L(p) \le \gamma\}, \sup\{p|\Psi_U(p) \le \gamma\}] \tag{29}$$

is defined as bounded value-at-risk.

The defined average risk is a reflection of the average belief degree of the risk index p, and a greater value of $\overline{p}$ means more severe risk that we believe we will suffer. The meaning of the bounded value-at-risk is that, with belief degree γ, we believe that the value of risk index is within the interval $[\text{VaR}_L, \text{VaR}_U](\gamma)$. Obviously, if we fix the value of γ, a wider bounded value-at-risk means a more conservative assessment result. Meanwhile, we believe a greater $\text{VaR}_U(\gamma)$ reflects that the risk is more severe.

Table 3 Distributions of level-1 and level-2 parameters

Parameter	Level-1	Level-2 UTB Method	Level-2 ETB Method
x_1	$N(\mu_1, 5)$	$\mu_1 \sim \mathcal{L}(9, 11)$	$\mu_1 \sim U(9, 11)$
x_1	$N(\mu_2, 5)$	$\mu 2 \sim \mathcal{N}(10, 0.3)$	$\mu_2 \sim N(10, 0.3)$

3.3.2 Numerical Case Study

We consider a problem of structural reliability in Choi et al (2007) to further elaborate on the developed method. Let the limit-state function of a structure be

$$g(x_1, x_2) = x_1^4 + 2x_2^4 - 20. \tag{30}$$

where x_1 and x_2 are random variables, and the risk index of interest is the probability that the structure fails, which can be written as

$$p_f = \Pr\{g(x_1, x_2) < 0\}. \tag{31}$$

Assume that x_1 and x_2 follow normal distributions with parameters (μ_1, σ_1) and (μ_2, σ_2), respectively. The parameters μ_1 and μ_2 are not precisely known due to the epistemic uncertainty, whereas σ_1 and σ_2 are known as crisp values. Based on experts knowledge, the belief degree of μ_1 is modeled by a linear uncertainty distribution and μ_2 is described by a normal uncertainty distribution (see Table 3). The bounded uncertainty distribution can, then, be obtained through Algorithm 1.

The solid line and dashed line in Figure 3 show the upper and lower uncertainty distributions of the risk index p_f, respectively. Average risk and bounded value-at-risk are calculated using the numerical method based on (28) and (29), i.e., $\overline{p} = 0.001980$ and $[\text{VaR}_L, \text{VaR}_U](0.9) = [0.001689, 0.003548]$.

Since the developed method offers a bounded uncertainty distribution of p_f, it is then compared with an evidence theory-based (ETB) method, in which the belief degree of p_f is also given as upper and lower distributions called plausibility (Pl) and belief (Bel) function, respectively. In this paper, the ETB method models the belief degrees of μ_1 and μ_2 using probability distributions (see Table 3).A double loop Monte Carlo simulation combined with a discretization method for getting basic probability assignments (BPAs) is used to obtain $Bel(p_f)$ and $Pl(p_f)$ (Limbourg and Rocquigny, 2010; Tonon, 2004). In Figure 3, the dotted line and dot-dash line represent Bel and Pl, respectively. It should be noted that although we use Bel and Pl as mathematical constructs, they are not strictly the concepts of Belief and Plausibility defined by Shafer (i.e. the degree

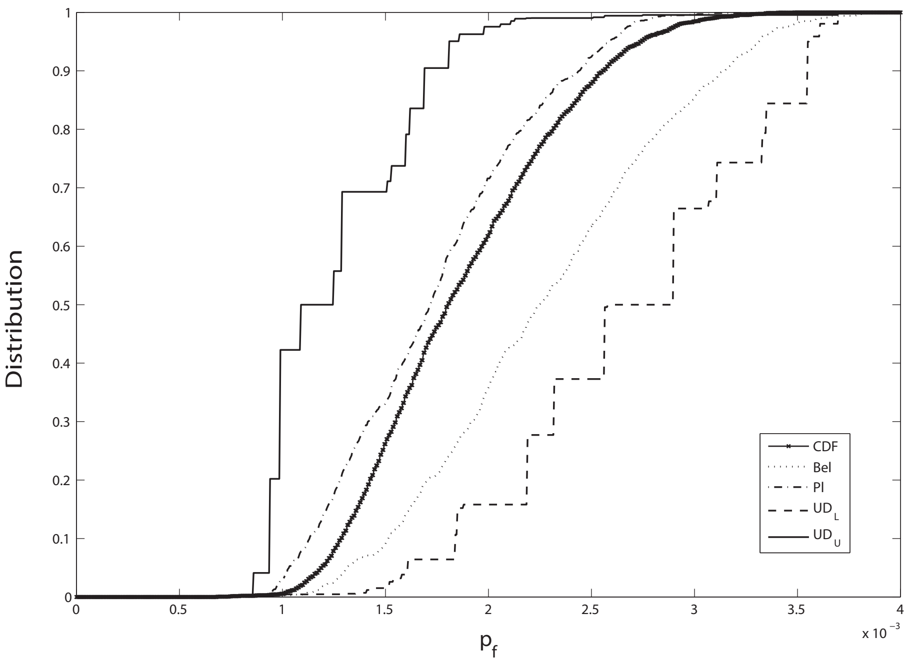

Fig. 3 Results of level-2 uncertainty analysis (CDF: cumulative distribution function, *Bel*: belief function, *Pl*: plausibility function, UD_L: lower uncertainty distribution, UD_U: upper uncertainty distribution.)

Table 4 Risk indexes of the non-monotone risk model

Method	$\overline{p}$	$[\text{VaR}_L, \text{VaR}_U]\,(0.9)$
UTB Method	0.001980	[0.001689,0.003548]
ETB Method	0.002012	[0.002440,0.003140]

of truth of a proposition (Shafer, 1976)). The two functions only represent bounds on a true quantity. To illustrate this, a cumulative density function (CDF) of p_f is calculated via a double loop MC simulation method, shown as the crossed line in Figure 3. It is seen that the CDF is covered by the area enclosed by *Bel* and *Pl*. In this sense, the CDF obtained in PB method is a special case of the ETB model, and the $Bel\,(p_f)$ and $Pl\,(p_f)$ give a reasonable bound of the probability distribution of p_f.

Given $Bel\,(p_f)$ and $Pl\,(p_f)$, the two risk indexes can be calculated by

$$\overline{p} = \int_0^{\infty} \left(1 - \frac{Bel(p_f) + Pl(p_f)}{2}\right) dp, \tag{32}$$

and

$$[\text{VaR}_L, \text{VaR}_U]\,(\gamma) = [\sup\{p_f | Bel(p_f) \le \gamma\}, \sup\{p_f | Pl(p_f) \le \gamma\}], \tag{33}$$

and the results are tabulated in Table 4.

Figure 3 shows a comparison of the distributions of belief degrees on p_f in UTB method and ETB method. The distributions have the same supports, whereas the upper and lower uncertainty distributions fully cover the CDF and the area enclosed by *Bel* and *Pl*, which indicates that the developed method is more conservative. This is because the subjective belief described by uncertainty distributions usually tends to be more conservative, and is more easily affected by epistemic uncertainty. This phenomenon is also reflected by the two defined risk indexes: the average risk $\overline{p}$s are nearly the same on different theory basis, while the bounded value-at-risk of ETB method is within that of the UTB method.

We also find that the bounded value-at-risk obtained by the developed UTB method may be too wide for some decision makers. This may be a shortcoming of the proposed method. Therefore, when choosing a method for risk analysis from the PB method, ETB method and UTB method, we need to consider the attitude of decision maker. For a conservative decision maker, the bounded uncertainty distribution is an alternative choice.

4 Application

In this section, the developed level-2 uncertainty analysis method is applied to a real application of flood risk assessment. In Section 4.1, we briefly introduce the system of interest. Sections 4.2 and 4.3 show the process of level-2 uncertainty analysis based on uncertainty theory, to illustrate the effectiveness of the method.

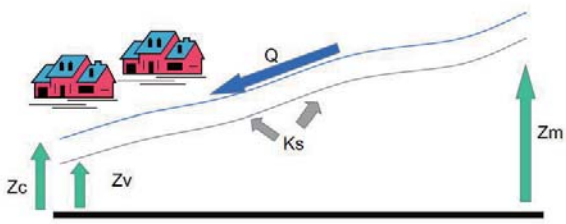

Fig. 4 Flooding risk system (Limbourg and Rocquigny, 2010)

4.1 System Description

In this case study, we consider a residential area located near a river, which is subject to potential risks of floods, as shown in Figure 4. As a mitigation and prevention measure, a dike is constructed to protect this area. The final goal is to calculate the risk of floods to determine whether the dike needs to be heightened. A mathematical model is develop in (Limbourg and Rocquigny, 2010) which calculates the maximal water level Z_c:

$$Z_c = g(Q, K_s, Z_m, Z_v, l, b) = Z_v + \left(\frac{Q}{K_s \cdot b \cdot \sqrt{(Z_m - Z_v)/l}}\right)^{3/5}, \quad (34)$$

where Z_m denotes the riverbed level at the upstream part of the river, Z_v denotes the riverbed level at the downstream part of the river, K_s denotes the friction coefficient of the riverbed, Q denotes the yearly maximal water flow, l denotes the length of river, and b denotes the width of river (Limbourg et al, 2010). The risk of floods can, then, be calculated as the probability that the annual maximum water level exceeds the dike height:

$$p_{\text{flood}} = \Pr\{Z_c > H\}. \quad (35)$$

4.2 Parameter Setting

The input variables in 34 are assumed to be random variables and the form of their PDFs are assumed to be known, as shown in Table 5 (Limbourg and Rocquigny, 2010). However, due to limited statistical data, the distribution parameters of these PDFs cannot be precisely estimated using statistic methods and therefore, are affected by epistemic uncertainty, which should be evaluated based on experts knowledge. In this paper, the experts knowledge on the distribution of these parameters is obtained by asking the experts to give the uncertainty distributions of the parameters, as shown in Table 5.

Table 6 Risk indexes for the flood system

Index	Value
$\overline{p_{\text{flood}}}$	0.0161
$[\text{VaR}_L, \text{VaR}_U]\,(0.9)$	[0.0073,0.0476]

For example, the yearly maximal water flow, denoted by Q, follows a Gumbel distribution $Gum(\alpha, \beta)$, and according to expert judgements, α and β follow normal uncertainty distributions $\mathcal{N}_\alpha(1013, 48)$ and $\mathcal{N}_\beta(558, 36)$, respectively. In addition, considering some physical constraints, the input quantities also have theoretical bounds, as given in Table 5.

4.3 Results and Discussions

Uncertain simulation method is used to propagate the level-2 uncertainty using Algorithm 1. The theoretical bounds in Table 5 are considered by truncating the probability distributions at these bounds. The lower and upper bounds for the uncertainty distributions of p_{flood} are shown in Figure 5, which represents the belief degrees on p_{flood} considering the level-2 uncertainty. Average risk and bounded value-at-risk are calculated based on (28) and (29) and presented in Table 6.

It follows that the average yearly risk is p_{flood}, which corresponds to an average return period of 62 years. This is unacceptable in practice, because it is too short when compared to a commonly required 100-year-return period. To solve this problem, one measure is to heighten the dike for a more reliable protection. Another solution might be increasing the friction coefficient of the riverbed K_s, noting from 34 that Z_c decreases with K_s.

The bounded Value-at-risk is relatively wide, which indicates that due to the presence of epistemic uncertainty, we cannot be too confirmed on the calculated risk index. The same conclusion can also be drawn from Figure 5: the difference between the upper and lower bounds of the uncertainty distributions are large, indicating great epistemic uncertainty. To reduce the effect of epistemic uncertainty, more historical data need to be collected to support a more precise estimation of the distribution parameters in the level-1 probability distributions.

5 Conclusions

In this paper, a new level-2 uncertainty analysis method is developed based on uncertainty theory. The method is discussed in two respects: for monotone risk models, where the risk index of interest is expressed

Table 5 Uncertainty description of level-1 and level-2 parameters

Parameter	Probability distribution	Level-2 uncertainty distribution	Theoretical bounds
Q	$Gum(\alpha, \beta)$	α $\mathcal{N}_\alpha(1013, 48)$ β $\mathcal{N}_\beta(558, 36)$	[10,10000]
K_s	$N(\mu_{K_s}, \sigma^2_{K_s})$	μ_{K_s} $\mathcal{L}(22.3, 33.3)$ σ_{K_s} $\mathcal{L}(2.5, 3.5)$	[5,60]
Z_m	$N(\mu_{Z_m}, \sigma^2_{Z_m})$	μ_{Z_m} $\mathcal{L}(54.87, 55.19)$ σ_{Z_m} $\mathcal{L}(0.33, 0.57)$	[53.5,57]
Z_v	$N(\mu_{Z_v}, \sigma^2_{Z_v})$	μ_{Z_v} $\mathcal{L}(50.05, 50.33)$ σ_{Z_v} $\mathcal{L}(0.28, 0.48)$	[48,51]
l	5000(constant)		-
b	30(constant)		-

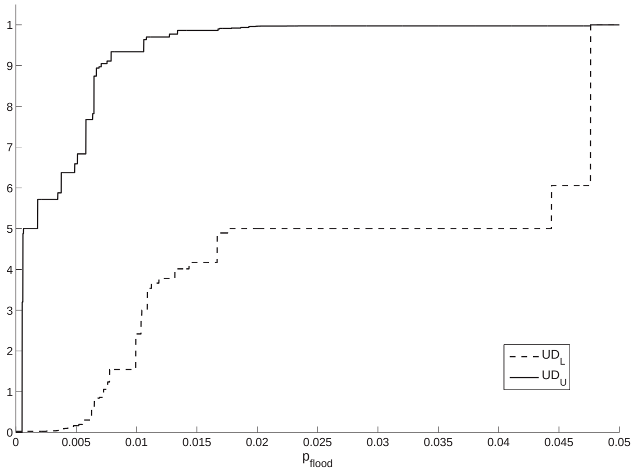

Fig. 5 Result for level-2 uncertainty propagation based on UTB method

as an explicit monotone function of the uncertain parameters, and level-2 uncertainty analysis is conducted based on operational laws of uncertainty variables; for non-monotone risk models, an uncertain simulation-based method is developed for level-2 uncertainty analysis. Three indexes, i.e., average risk, Value-at-risk and bounded Value-at-risk, are defined for risk-informed decision making in the level-2 uncertainty setting. Two numerical studies and an application on a real example from literature are worked out to illustrate the developed method. The developed method is also compared to some commonly-used level-2 uncertainty analysis methods, e.g., PB method and ETB method. The comparisons show that, in general, the UTB method is more conservative than the methods based on probability theory and evidence theory.

6 Compliance with Ethical Standards

The authors declare that there is no conflict of interest regarding the publication of this paper. This article does not contain any studies with human participants performed by any of the authors.

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Optimization of Spare Parts Varieties Based on Stochastic DEA Model

Meilin Wen, Tianpei Zu, Miaomiao Guo, Rui Kang* Yi Yang

***Abstract*—Accurate inventory management starts with the scientific and rational classification of numerous varieties of spare parts. This paper presents a stochastic data envelopment analysis (DEA) model to address the problem of optimization of spares varieties under uncertainty. An index system is proposed in terms of product life-cycle process, which contains five design indexes, four operation indexes and five support indexes. Then quantification method of the index system is briefly discussed in preparation for mathematical calculation. A stochastic spares optimization model (SSOM) is proposed based on stochastic DEA with the constraints of 14 factors of the index system. The SSOM could be converted into equivalent deterministic models by probability theory, which overcomes the difficulty in solving non-linear programming. A numerical example is given to illustrate the proposed method in terms of ability to provide reasonable inventory management policies.**

***Index Terms*—uncertain environment, spare parts, optimization, stochastic, data envelopment analysis**

I. Introduction

SPARES parts, which are stocked to replace failed parts, are the indemnification goods for plants to maintain normal functioning. Stocking strategy for spare parts has a pivotal influence on the productivity and efficiency of industrial plants. Conservative strategy may lead to overstocking and high cost of inventory, which will reduce the profit of industrial plants. On the other hand, optimistic strategy may result in shortage of necessary spare parts, long machine downtime and decrement in productivity. Therefore, optimization of spare parts varieties plays an important role in inventory management.

There are considerable existing literatures on optimizing stocking strategy of spare parts. It is important to have a reasonable index system before optimizing the controlling strategy. Some researchers tried to solve this problem by a single attribute. For example, Nahmias [1] proposed a one-dimensional approach that is only based on total cost. However, strategy with only one single attribute cannot solve the problem when the spare parts are competing. Many multi-attribute evaluation systems were then developed to overcome this difficulty. As was reported in Ng [2], Zhou and Fan [3], Hadi-Vencheh [4] and Lin et al. [5], a classification scheme including annual dollar usage, lead time and average unit cost was proposed. Multi-attribute decision making techniques were employed by Molenaers [6], Almeida [7] and Sharaf and Helmy [8] to provide reasonable decision-making proposals. Deterministic attribute models were gradually developed into random attribute models in terms of parameter dispersion. Quantitative analyses for indeterministic variables were then carried out. Wang [9] established a stochastic model for joint spare parts inventory and planned maintenance optimization considering the random nature of plant failures and then applied stochastic dynamic programming to find the optimal solutions over a finite time horizon. Godoy et al. [10] presented a graphical technique which considered a stochastic lead time and a reliability threshold to enhance spare parts ordering decision-making. Gu et al. [11] assumed that the probability density distribution functions of lifetime and the number of failures follow normal distributions, then worked out the optimal order quantities by minimizing the total cost. Li et al. [12] proposed a stochastic programming model to seek a optimal spare parts ordering and pricing policy from a distributor's view. Zamar et al. [13] developed a quantile-based scenario analysis approach for stochastic supply chain optimization under uncertainty. However, the above research mainly focus on factors that influence the demand of spare parts in the normal operational stage or supporting stage and few articles investigate the factors in the whole product life-cycle process. In this paper, we will establish a comprehensive index system in terms of the whole product life-cycle process, including design factors, operation factors and support factors.

Once the evaluation indexing system is established, it will need to choose a kind of evaluation method that can evaluate the importance of these factors. Many methods have emerged in this field. The well-known ABC classification [3], [4], [14] is simple to use and easy to understand. However, ABC analysis is based on only single measurement such as annual dollar usage. Other criteria have gradually been recognized to be important in inventory management. Analytic hierarchy process (AHP) was adopted to determine the weights of factors for multi-attribute evaluation system. For example, Braglia [15] employed AHP on an inventory policy matrix based on the reliability centered maintenance (RCM) to identify the best control strategy. Molenaers [6] firstly presented the multi-criteria classification issue in a logic decision tree based

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on item criticality, then used AHP at different nodes of the diagram and converted relevant criteria into a single scalar to represent the criticality of the part. However, AHP requires subjective judgment when making pair-wise comparisons. Heuristic algorithms like genetic algorithms [16], [17] and artificial neural networks [18], [19] were also utilized to evaluate the importance of index system. However, they are complex and difficult in application.

Data Envelopment Analysis (DEA) is a Linear Programming based technique for the analysis of efficiency of organizations with multiple inputs and outputs and is proposed by Charnes et al. in 1978 [20]. In DEA, the organization under study is called a DMU (decision-making unit). Suppose there are n DMUs in a DEA model: DMU_0 is target DMU, DMU_i is the ith DMU ($i = 1, 2, \cdots, n$), $\boldsymbol{x_i} = (x_{i1}, x_{i2}, \cdots, x_{ip})$ is the inputs vectors of DMU_i, $\boldsymbol{x_0} = (x_{01}, x_{02}, \cdots, x_{0p})$ is the inputs vector of DMU_0, $\boldsymbol{y_i} = (y_{i1}, y_{i2}, \cdots, y_{ip})$ is the outputs vectors of DMU_i, $\boldsymbol{y_0} = (y_{01}, y_{02}, \cdots, y_{0p})$ is the outputs vector of DMU_0, $\boldsymbol{u} \in \boldsymbol{R}^{p\times 1}$ is the vector of input weights, $\boldsymbol{v} \in \boldsymbol{R}^{p\times 1}$ is the vector of output weights. Then a typical basic DEA model called CCR is represented as follows:

$$\begin{cases} \max\ \theta = \frac{\boldsymbol{v^t y_0}}{\boldsymbol{u^t x_0}} \\ subject\ to: \\ \boldsymbol{v^t y_i} \le \boldsymbol{u^t x_i}, i = 1, 2, \cdots, n, \\ \boldsymbol{u} \ge 0 \\ \boldsymbol{v} \ge 0. \end{cases} \tag{1}$$

In the above model, the objective is to obtain the ratio of the weighted output to the weighted input weights with the constraints that the ratio of virtual output vs. virtual input should not exceed 1 for every DMU. By virtue of the constraints, the optimal objective value θ^* is at most 1. The optimal solution θ^* yields an efficiency score for a particular DMU and the process is repeated for each DMU_i, $i = 1, 2, \cdots, n$. DEA method can be regarded as a production process with multiple inputs and outputs. As is known, all the manufactures hope to produce maximum outputs with the least inputs. This principle is also reflected in DEA model. The DMU will be more effective than other DMUs if it has a larger optimal value. Therefore, DMUs are regarded to be inefficient if $\theta^* < 1$, while DMUs are efficient if $\theta^* = 1$. Compared with the aforesaid methodology, DEA have the advantages in avoiding subjective factors, having simple algorithms and reducing errors. Moreover, it can contain controllable input (output) and non-controllable input (output). DEA can efficiently deal with fact that the numerical dimension is not unified. Several DEA models, i.e. BCC model and Additive model, have been developed to suit different application scenarios [21]–[24]. In view of the situation that inputs and outputs of the DMU cannot be accurately determined, many literatures have proposed opportunity constrained programming models [25]–[27], stochastic DEA models [28]–[31] and fuzzy DEA models [32]. In this paper, some factors in index system, i.e. corrective maintenance time, logistic delay time and mission time, are assumed to be random variables, considering that their values will change with real-life situations. Therefore, a stochastic DEA model is employed to address the problem of optimization of spare parts varieties.

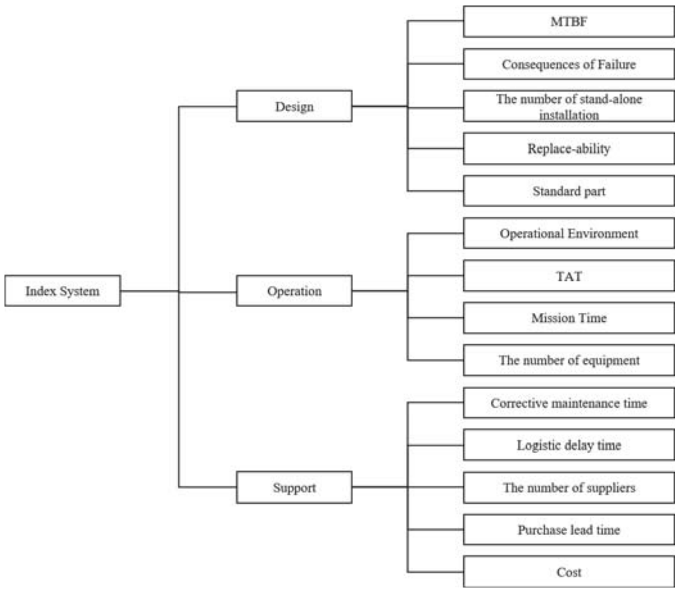

Fig. 1. Index system

The reminder of this paper is organized as follows. In section 2, a comprehensive index system is established in terms of the whole product life-cycle process. Section 3 briefly introduces the quantification method of the index systems with respect to qualitative factors and quantitative factors. Section 4 establishes the stochastic spares optimization model (SSOM) based on stochastic DEA method. Some algorithms are addressed to obtain the equivalent deterministic model in Section 5. A numerical example is given to illustrate the SSOM in Section 6. Section 7 summarizes the main work and contributions of this paper.

II. Establishment of Index System

In this section, an index system is established in terms of design, operation and support as shown in Fig. 1. Design factors are composed of mean time between failures (MTBF), consequences of failure (CoF), the number of stand-alone installation, replace-ability and standard part, which are determined in the design phase. Operation factors are the attributes that influence the inventory of spare parts in the operation phase, including operation environment, turn around time (TAT), mission time and the number of equipment. Support factors consist of corrective maintenance time, logistic delay time, the number of suppliers, purchase lead time and cost, which are related to logistics and maintenance.

A. Design indexes

The subsection below describes the properties of five design indexes.

MTBF refers to the average amount of time that a device or product functions before failing, which is an important index in repairable system [33]. MTBF is an important parameter for measuring the reliability and availability [34], [35]. Spare parts with short MTBFs are always at a low level of reliability and availability and need frequent maintenance. Therefore, items with short MTBFs should be kept at high inventory levels.

CoF refers to losses or damages that are caused by a failure. Generally, consequences of failure could be classified into five types in terms of severity, i.e. catastrophic consequences, critical consequences, severe consequences, marginal consequences and negligible consequences. Items that are likely to cause serious consequences may high inventory levels because their shortage of spare parts will have a critical impact on the overall system.

The number of stand-alone installation is the number of a kind of component or unit installed on an equipment, which is clarified in the design phase. The number of stand-alone installation may, to some extent, reflect the demand for spare parts. Components with a large number of stand-alone installation are more likely to fail, so the demand for spare parts is higher.

Replace-ability is the capability of an item to be replaced by site workers. Components without replace-ability cannot be replaced in the current site and need to be sent to senior site. Thus, it is reasonable to reduce the inventory level of items with poor site replace-ability.

A standard part refers to a part or material that conforms to an established industry published specification. The lack of non-standard parts is more difficult to handle than standard parts because standard parts are interchangeable and easier to obtain than non-standard parts. Therefore, it is necessary to maintain the inventory level of non-standard parts.

B. Operation indexes

This subsection gives a brief introduction of operation indexes, including mission time, operation environment, turnaround time (TAT) and the number of equipment.

Mission refers to the task, together with the purpose, that clearly indicates the action to be taken and the reason therefore [36]. Mission time is the length of time to complete the mission. Mission time has a direct impact on the demand for spare parts because more spare parts may need to be replaced within a longer mission time.

Operational environment is defined as a composite of the conditions, circumstances, and influences that affect the employment of capabilities and bear on the decisions of the commander [36]. The operating environment has a crucial influence on the life of components and the demand for components. For example, the operating environment of aircraft engines is more stringent than displays installed in cockpits in the same plane. The screws on the engine are more likely to fail than the screws on the display. Therefore, the spare parts for the screws on the engine are higher than those on the display.

Turnaround time (TAT) is defined as the length of time between arriving at a point and being ready to depart from that point [36]. TAT depends on the properties of equipment as well as maintainability and supportability in practice, and will change with the real scenarios. TAT has a obvious influence on the requirement of spare parts as there is not enough time to prepare spare parts in case of limited TAT.

The sum of equipments is the total number of the equipments participating in the mission. The sum of equipments is of interest because the requirement of inventory of spare parts is in a proportional relationship with it.

C. Support Indexes

The subsection below describes the properties of five support indexes.

Corrective maintenance time is the time that begins with the observance of a malfunction of an item and ends when the item is restored to a satisfactory operating condition [37]. Corrective maintenance time could represent one item's maintainability. Components with long corrective maintenance time are poor in maintainability and thus should maintain high inventory levels.

Logistic delay time is the component of downtime during which no maintenance is being accomplished on the item because of technician alert and response time, supply delay, or administrative reasons [37]. As the shortage of these spare parts may cause huge time and money costs, spare parts with long logistics delay time should maintain a high inventory level.

Purchase lead time is defined as the time between the initiation and completion of a purchase process and could be determined by market research and historical experience. Parts with long purchase lead time are more likely to be in short supply. Therefore, spare parts with a long procurement cycle should maintain a stable inventory level.

The number of suppliers is the sum of suppliers who can provide required spare parts. According to MIL-STD-965B, components are selected from program parts selection list (PPSL) and their suppliers are then determined. The number of suppliers has a direct effect on the supply stability because components with the large number of suppliers are at a low risk in shortage.

Cost include cost of purchase and cost of storage. The cost of spare parts is determined by their natural properties and is therefore determined during the design phase. It is intuitive to store the spare parts whose cost is at a low level with respect to costs-saving.

III. Quantification method of index system

Quantification is an important step before we take the indexes into mathematical models. Based on the properties of indexes demonstrated in Section II, these indexes could be classified into two types, qualitative indexes and quantitative indexes.

Qualitative indexes include operational environment (OE), replace-ability (RA), standard part (SP), consequences of failure (CoF). To employ these factors in the mathematical model, Table I shows the qualification principles of these indexes.

Quantitative indexes could be further divided into two subtypes, deterministic factors and random factors. Deterministic factors could be represent by crisp values and include the sum of equipments, the number of stand-alone installation, purchase lead time, the number of suppliers and cost. Specifically, purchase lead time, the number of suppliers and cost could obtained by market research or historical experiences. The sum of equipments and the number of stand-alone installation are

TABLE I
QUANTIFICATION PRINCIPLES

	1	3	5	7	9
OE	Particularly harsh	Severe	Moderate	Mild	Gentle
CoF	Catastrophic	Critical	Severe	Marginal	Negligible
RA	Cannot be replaced	Can be replaced			
SP	Standard part	Non-standard part			

designed according to the requirements of mission. Random factors are the variables that will change with actual scenarios, including TAT, MTBF, mission time, corrective maintenance time and logistic delay time. The approach to determine these quantitative indexes is not demonstrated here as it is not the focus of this paper.

IV. Modeling SSOM Based on Stochastic DEA

In this section, we develop a stochastic programming model based on Additive model proposed by Charnes et al. [22] in 1985 which considers the total slacks of inputs and outputs simultaneously in arriving at a point on the efficient frontier. In the proposed stochastic spares optimization model (SSOM), every candidate inventory item is regarded as a decision-making unit (DMU). The constraints derive from the index system. The objective is to find the optimum which simultaneously maximizes outputs and minimizes inputs in the sense of vector optimizations. The candidate DMU is efficient when the objective is zero, based on which we can rank all candidate DMUs. The section is organized as follows. Firstly, we give a brief introduction on the relative symbols and notations. Then we classify 14 factors into input variables and output variables. The SSOM is subsequently established with input vectors and output vectors. Finally, a ranking criterion is given and illustrated, based on which we can give priorities to all candidate DMUs.

Assume that there are n DMUs and then relative symbols and notations are introduced as follows:

DMU_k : the kth DMU, $k = 1, 2, \cdots, n$;
DMU_0 : target DMU;
$\widetilde{T}_k$: the random TAT of DMU_k;
$\widetilde{F}_k$: the random MTBF of DMU_k;
$\widetilde{W}_k$: the random mission time of DMU_k;
$\widetilde{M}_k$: the random corrective maintenance time of DMU_k;
$\widetilde{D}_k$: the random logistic delay time of DMU_k;
S_k : the number of suppliers of DMU_k;
E_k : the operational environment of DMU_k
G_k : the consequences of failure of DMU_k;
A_k : the replace-ability of DMU_k;
P_k : standard part or not of DMU_k;
N_k : the number of stand-alone installation of DMU_k;
Z_k : the number of equipments of DMU_k;
L_k : the purchase lead time of DMU_k;
C_k : the cost of DMU_k;
λ_k: the weght of kth DMU $i = 1, 2, \cdots, n$;
s_i^-: the slack of each ith input;
s_j^+: the slack of each jth output;
$\Pr$ is the probability measure;
α is belief degree which is a predetermined number between 0 and 1.

DEA method can be regarded as a production process with multiple inputs and outputs. As is known, all the manufactures hope to produce maximum outputs with the least inputs. Therefore, this principle is also reflected in DEA model. The DMU will be more effective than other DMUs if it has a smaller input as well as a larger output. According to the strategy tendency with smaller inputs and larger outputs, we divide all the parameters into input indexes and output indexes. It should be classified as input index if more attentions need to be paid to the smaller parameter, conversely it should be regarded as output index. For example, MTBF is regarded as an input index because items with shorter MTBF are more important with the aim of selecting pivotal spare parts varieties. By contrast, purchase lead time is regarded as an output variable as it is more likely to be short of the spare parts whose purchase lead time is long. The inputs and outputs are:

$X_k = \{\widetilde{T}_k, \widetilde{F}_k, S_k, E_k, G_k, A_k, C_k\}, \quad k = 1, 2, 3, \cdots, n;$
$Y_k = \{\widetilde{W}_k, N_k, Z_k, \widetilde{M}_k, L_k, \widetilde{D}_k, P_k\}, \quad k = 1, 2, 3, \cdots, n.$

Then the SSOM is:

$$\begin{cases} \max\ \theta = \sum\limits_{i=1}^{7} s_i^- + \sum\limits_{j=1}^{7} s_j^+ \\ subject\ to: \\ \Pr\left\{ \sum\limits_{k=1}^{n} \widetilde{T}_k \lambda_k \le \widetilde{T}_0 - s_1^- \right\} \ge \alpha, \\ \Pr\left\{ \sum\limits_{k=1}^{n} \widetilde{F}_k \lambda_k \le \widetilde{F}_0 - s_2^- \right\} \ge \alpha, \\ \Pr\left\{ \sum\limits_{k=1}^{n} \widetilde{W}_k \lambda_k \ge \widetilde{W}_0 + s_1^+ \right\} \ge \alpha, \\ \Pr\left\{ \sum\limits_{k=1}^{n} \widetilde{M}_k \lambda_k \ge \widetilde{M}_0 + s_4^+ \right\} \ge \alpha, \\ \Pr\left\{ \sum\limits_{k=1}^{n} \widetilde{D}_k \lambda_k \ge \widetilde{D}_0 + s_6^+ \right\} \ge \alpha, \\ \sum\limits_{k=1}^{n} S_k \lambda_k \le S_0 - s_3^-, \\ \sum\limits_{k=1}^{n} E_k \lambda_k \le E_0 - s_4^-, \\ \sum\limits_{k=1}^{n} G_k \lambda_k \le G_0 - s_5^-, \\ \sum\limits_{k=1}^{n} A_k \lambda_k \le A_0 - s_6^-, \\ \sum\limits_{k=1}^{n} C_k \lambda_k \le C_0 - s_7^-, \\ \sum\limits_{k=1}^{n} N_k \lambda_k \ge N_0 + s_2^+, \\ \sum\limits_{k=1}^{n} Z_k \lambda_k \ge Z_0 + s_3^+, \\ \sum\limits_{k=1}^{n} L_k \lambda_k \ge L_0 + s_5^+, \end{cases} \tag{2a}$$

$$\begin{cases} \sum_{k=1}^{n} P_k\lambda_k \geq P_0 + s_7^+, \\ \sum_{k=1}^{n} \lambda_k = 1, \\ \lambda_k \geq 0, \quad k = 1, 2, \cdots, n \\ s_i^- \geq 0, \quad i = 1, 2, \cdots, 7 \\ s_j^+ \geq 0, \quad j = 1, 2, \cdots, 7 \end{cases} \tag{2b}$$

Ranking criterion: The closer θ is to zero, the more efficient the DMU_0 is ranked.

The θ is the sum of all the input slacks and output slacks for one DMU. To let θ close to 0, either the inputs are small, or the outputs are larger, or both of them. Thus the closer to 0 the θ is, the more potential the DMU has got to be reserved.

We can give priorities to DMUs by the ranking criterion, based on which inventory policies could be determined.

V. Equivalent Deterministic Model

This section simplifies the constraints of random variables and develops equivalent deterministic models to overcome the difficulty in solving nonlinear programming.

Defination 1. *(Liu [38]) Suppose that ξ is a random variable defined on probability space $(\Omega, \tilde{A}, \Pr)$. For any $\alpha \in (0, 1]$, the α-optimistic values of ξ are defined as*

$$\xi_{\sup}(\alpha) = \sup\{r | \Pr\{\xi \geq r\} \geq \alpha\}.$$

Defination 2. *(Liu [38]) Suppose that ξ is a random variable defined on probability space $(\Omega, \tilde{A}, \Pr)$. For any $\alpha \in (0, 1]$, the α-pessimistic values of ξ are defined as*

$$\xi_{\inf}(\alpha) = \inf\{r | \Pr\{\xi \leq r\} \geq \alpha\}.$$

Defination 3. *A real-valued function f defined on a convex set $X \in R^n$ is said to be quasiconcave if*

$$f(\lambda x + (1-\lambda) v) \geq \min\{f(x), f(v)\}$$

for any $x, y \in X$ and $0 < \lambda < 1$.

Theorem 1. *Assume $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n$ are independent random variables defined on probability space $(\Omega, \tilde{A}, \Pr)$. If $\Pr\{\tilde{T}_k = x_k\}(k = 1, 2, \cdots, n)$ are quasiconcave, and any α is given in $(0.5, 1]$, $\lambda_k \in [0, 1]$, then for*

$$\Pr\left\{\sum_{k=1}^{n} \widetilde{T}_k\lambda_k \leq \widetilde{T}_0 - s_1^-\right\} \geq \alpha, \tag{3}$$

we have

$$\sum_{k=1, k\neq 0}^{n} \{\lambda_k(\widetilde{T}_k)_{\inf}(\alpha)\} + \lambda_0[(\widetilde{T}_0)_{\sup}(\alpha)] \leq (\widetilde{T}_0)_{\sup}(\alpha) - s_1^-. \tag{4}$$

Proof. *Without loss of generality, let $n = 2$, $\lambda_1 = \lambda_0$ and $\tilde{T}_1 = \tilde{T}_0$, then we will consider the equation*

$$\Pr\left\{\widetilde{T}_0\lambda_0 + \widetilde{T}_2\lambda_2 \leq \widetilde{T}_0 - s_1^-\right\} \geq \alpha. \tag{5}$$

Rewrite equation (5) as

$$\Pr\left\{(1-\lambda_0)\,\widetilde{T}_0 + (-\lambda_2)\,\widetilde{T}_2 \leq s_1^-\right\} \leq 1 - \alpha. \tag{6}$$

Then we have

$$\begin{aligned} &\Pr\{(1-\lambda_0)\,\widetilde{T}_0 + (-\lambda_2)\,\widetilde{T}_2 \leq s_1^-\} = \\ &1 - \sup_{x_1+x_2>s_1^-} \{\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1\} \wedge \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2\}\} \\ &\leq 1-\alpha. \end{aligned} \tag{7}$$

Hence,

$$\sup_{x_1+x_2>s_1^-} \{\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1\} \wedge \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2\}\} \geq \alpha.$$

Suppose that $(x_1^*, x_2^*) = \arg \sup_{x_1+x_2 \in R} \Pr\{\{(1-\lambda_0)\,\widetilde{T}_0 = x_1\} \wedge \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2\} | \{x_1 + x_2 > s_1^-\}\} \geq \alpha\}$.

It follows that $\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1^*\} \wedge \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2^*\} \geq \alpha$ and $x_1^* + x_2^* > s_1^-$.

Since $\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1^*\} \wedge \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2^*\} \geq \alpha$ implies that $\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1^*\} \geq \alpha$, $\Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2^*\} \geq \alpha$.

From that the functions $\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1\}$ and $\Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2\}$ are quasiconcave, we have

$$x_1^* \leq ((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha),\ x_2^* \leq ((-\lambda_2)\,\widetilde{T}_2)_{\sup}(\alpha).$$

Then we get

$$((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha) + ((-\lambda_2)\,\widetilde{T}_2)_{\sup}(\alpha) \geq s_1^-.$$

Otherwise,

$$((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha) + ((-\lambda_2)\,\widetilde{T}_2)_{\sup}(\alpha) < s_1^-,$$

$\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = ((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha)\} \leq \Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1^*\}$,
$\Pr\{(-\lambda_2)\,\widetilde{T}_2 = ((-\lambda_2)\,\widetilde{T}_2)_{\sup}(1-\alpha)\} \leq \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2^*\}$,
which are contradict with probability function $\Pr\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$.

Conversely, if

$$((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha) + ((-\lambda_2)\,\widetilde{T}_2)_{\sup}(\alpha) \geq s_1^-,$$

we get

$$\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = a_1\} \geq \alpha,$$

$$\Pr\{(-\lambda_2)\,\widetilde{T}_2 = a_2\} \geq \alpha.$$

since $a_1 < ((1-\lambda_0)\,\widetilde{T}_0)_{\sup}\alpha$, $a_2 < ((-\lambda_2)\,\widetilde{T}_2)_{\sup}(\alpha)$.

Consequently, $\sup_{x_1+x_2 \in R} \Pr\{\{(1-\lambda_0)\,\widetilde{T}_0 = x_1\} \wedge \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2\} | \{x_1 + x_2 > s_1^-\}\} \geq \alpha\}$.

Then,

$$\begin{aligned} &\Pr\{(1-\lambda_0)\,\widetilde{T}_0 + (-\lambda_2)\,\widetilde{T}_2 \leq s_1^-\} = \\ &1 - \sup_{x_1+x_2>s_1^-} \{\Pr\{(1-\lambda_0)\,\widetilde{T}_0 = x_1\} \wedge \Pr\{(-\lambda_2)\,\widetilde{T}_2 = x_2\}\} \\ &\leq 1-\alpha. \end{aligned}$$

Finally, we can get

$$((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha) + (\sum_{k=1, k\neq 0}^{n} \Big((-\lambda_k)\,\widetilde{T_k})_{\sup}(\alpha)\Big) \geq s_1^- \tag{8}$$

If $\lambda_k = 0$ or 1, it is obvious that

$$((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha) = (1-\lambda_0)\,(\widetilde{T}_0)_{\sup}(\alpha)$$
$$((-\lambda_k)\,\widetilde{T_k})_{\sup}(\alpha) = (-\lambda_k)\,\widetilde{(T_k)}_{\inf}(\alpha)$$

If $\lambda_k \in (0,1)$, then $1-\lambda_\theta > \theta$,$-\lambda_k < \theta$,

$$\begin{aligned}
&((1-\lambda_0)\,\widetilde{T}_0)_{\sup}(\alpha) = \sup\{r|\Pr\{\,(1-\lambda_0)\,\widetilde{T}_0 \ge r\} \ge \alpha\} \\
&= (1-\lambda_0)\sup\{r/\,(1-\lambda_0)\,|\Pr\{\widetilde{T}_0 \ge r/\,(1-\lambda_0)\} \ge \alpha\} \\
&= (1-\lambda_0)\,(\widetilde{T}_0)_{\sup}(\alpha), \\
&((-\lambda_k)\,\widetilde{T_k})_{\sup}(\alpha) = \sup\{r|\Pr\{\,(-\lambda_k)\,\widetilde{T_k} \ge r\} \ge \alpha\} \\
&= \ -\lambda_k \sup\{-r/\lambda_k|\Pr\{\widetilde{T_k} \le -r/\lambda_k\} \ge \alpha\} \\
&= (-\lambda_k)\,\widetilde{(T_k)}_{\inf}(\alpha).
\end{aligned}$$

Then, equation (8) can be rewritten as:

$$\sum_{k=1,k\neq 0}^{n} \{\lambda_k(\widetilde{T_k})_{\inf}(\alpha)\} + \lambda_0[(\widetilde{T}_0)_{\sup}(\alpha)] \le (\widetilde{T}_0)_{\sup}(\alpha) - s_1^-. \tag{9}$$

Similarly, we may simplify other random constraints in the same way, the model (2) can be rewrite as model (10).

$$\begin{cases}
\max\ \theta = \sum\limits_{i=1}^{7} s_i^- + \sum\limits_{j=1}^{7} s_j^+ \\
subject\ to: \\
\sum\limits_{k=1}^{n} \{\lambda_k(\widetilde{T}_k)_{\inf}(\alpha)\} + \lambda_0[(\widetilde{T}_0)_{\sup}(\alpha) - (\widetilde{T}_0)_{\inf}(\alpha)] \\
\le (\widetilde{T}_0)_{\sup}(\alpha) - s_1^-, \\
\sum\limits_{k=1}^{n} \{\lambda_k(\widetilde{F}_K)_{\inf}(\alpha)\} + \lambda_0[(\widetilde{F}_0)_{\sup}(\alpha) - (\widetilde{F}_0)_{\inf}(\alpha)] \\
\le (\widetilde{F}_0)_{\sup}(\alpha) - s_2^-, \\
\sum\limits_{k=1}^{n} \{\lambda_k(\widetilde{W}_k)_{\sup}(\alpha)\} + \lambda_0[(\widetilde{W}_0)_{\inf}(\alpha) - (\widetilde{W}_0)_{\sup}(\alpha)] \\
\ge (\widetilde{W}_0)_{\inf}(\alpha) + s_1^+, \\
\sum\limits_{k=1}^{n} \{\lambda_k(\widetilde{M}_k)_{\sup}(\alpha)\} + \lambda_0[(\widetilde{M}_0)_{\inf}(\alpha) - (\widetilde{M}_0)_{\sup}(\alpha)] \\
\ge (\widetilde{M}_0)_{\inf}(\alpha) + s_4^+, \\
\sum\limits_{k=1}^{n} \{\lambda_k(\widetilde{D}_k)_{\sup}(\alpha)\} + \lambda_0[(\widetilde{D}_0)_{\inf}(\alpha) - (\widetilde{D}_0)_{\sup}(\alpha)] \\
\ge (\widetilde{D}_0)_{\inf}(\alpha) + s_6^+, \\
\sum\limits_{k=1}^{n} S_k\lambda_k \le S_0 - s_3^-, \\
\sum\limits_{k=1}^{n} E_k\lambda_k \le E_0 - s_4^-, \\
\sum\limits_{k=1}^{n} G_k\lambda_k \le G_0 - s_5^-, \\
\sum\limits_{k=1}^{n} A_k\lambda_k \le A_0 - s_6^-, \\
\sum\limits_{k=1}^{n} C_k\lambda_k \le C_0 - s_7^-, \\
\sum\limits_{k=1}^{n} N_k\lambda_k \ge N_0 + s_2^+,
\end{cases} \tag{10a}$$

$$\begin{cases}
\sum\limits_{k=1}^{n} Z_k\lambda_k \ge Z_0 + s_3^+, \\
\sum\limits_{k=1}^{n} L_k\lambda_k \ge L_0 + s_5^+, \\
\sum\limits_{k=1}^{n} P_k\lambda_k \ge P_0 + s_7^+, \\
\sum\limits_{k=1}^{n} \lambda_k = 1, \\
\lambda_k \ge 0, \quad k = 1,2,\cdots,n \\
s_i^- \ge 0, \quad i = 1,2,\cdots,7 \\
s_j^+ \ge 0, \quad j = 1,2,\cdots,7
\end{cases} \tag{10b}$$

Especially, when random variables obey normal distributions and they are independent, the equivalent model can be rewrite as model (11),

$$\begin{cases}
\max\ \theta = \sum\limits_{i=1}^{7} s_i^- + \sum\limits_{j=1}^{7} s_j^+ \\
subject\ to: \\
\sum\limits_{k=1,k\neq 0}^{n} \overline{T}_k + \Phi^{-1}(\alpha)\sigma_{Tk}^I\lambda_k + \lambda_0[\overline{T}_0 - \sigma_{T0}^I\Phi^{-1}(\alpha)] \\
\le [\overline{T}_0 - \sigma_{T0}^I\Phi^{-1}(\alpha)] - s_1^-, \\
\sum\limits_{k=1,k\neq 0}^{n} \overline{F}_k + \Phi^{-1}(\alpha)\sigma_{Fk}^I\lambda_k + \lambda_0[\overline{F}_0 - \sigma_{F0}^I\Phi^{-1}(\alpha)] \\
\le [\overline{F}_0 - \sigma_{F0}^I\Phi^{-1}(\alpha)] - s_2^-, \\
\sum\limits_{k=1,k\neq 0}^{n} (\overline{W}_k - \Phi^{-1}(\alpha)\sigma_{Wk}^O)\,\lambda_k + \lambda_0[\overline{W}_0 + \sigma_{W0}^O\Phi^{-1}(\alpha)] \\
\ge [\overline{W}_0 + \sigma_{W0}^O\Phi^{-1}(\alpha)] + s_1^+, \\
\sum\limits_{k=1,k\neq 0}^{n} (\overline{M}_k - \Phi^{-1}(\alpha)\sigma_{Mk}^O)\,\lambda_k + \lambda_0[\overline{M}_0 + \sigma_{M0}^O\Phi^{-1}(\alpha)] \\
\ge [\overline{M}_0 + \sigma_{M0}^O\Phi^{-1}(\alpha)] + s_4^+, \\
\sum\limits_{k=1,k\neq 0}^{n} (\overline{D}_k - \Phi^{-1}(\alpha)\sigma_{Dk}^O)\,\lambda_k + \lambda_0[\overline{D}_0 + \sigma_{D0}^O\Phi^{-1}(\alpha)] \\
\ge [\overline{D}_0 + \sigma_{D0}^O\Phi^{-1}(\alpha)] + s_6^+, \\
\sum\limits_{k=1}^{n} S_k\lambda_k \le S_0 - s_3^-, \\
\sum\limits_{k=1}^{n} E_k\lambda_k \le E_0 - s_4^-, \\
\sum\limits_{k=1}^{n} G_k\lambda_k \le G_0 - s_5^-, \\
\sum\limits_{k=1}^{n} A_k\lambda_k \le A_0 - s_6^-, \\
\sum\limits_{k=1}^{n} C_k\lambda_k \le C_0 - s_7^-, \\
\sum\limits_{k=1}^{n} N_k\lambda_k \ge N_0 + s_2^+, \\
\sum\limits_{k=1}^{n} Z_k\lambda_k \ge Z_0 + s_3^+, \\
\sum\limits_{k=1}^{n} L_k\lambda_k \ge L_0 + s_5^+, \\
\sum\limits_{k=1}^{n} P_k\lambda_k \ge P_0 + s_7^+, \\
\sum\limits_{k=1}^{n} \lambda_k = 1,
\end{cases} \tag{11a}$$

$$\begin{cases} \lambda_k \geq 0, & k = 1, 2, \cdots, n \\ s_i^- \geq 0, & i = 1, 2, \cdots, 7 \\ s_j^+ \geq 0, & j = 1, 2, \cdots, 7 \end{cases} \quad (11b)$$

where $\Phi^{-1}(\alpha)$ is the inverse function of the standard normal distribution, σ_{TK}^I is the standard deviation of $T_k\,(k = 1, 2, ..., n)$, $\sigma_{T_0}^I$ is the standard deviation of T_0 , $\overline{T}_k$ is the average value of $T_k\,(k = 1, 2, ..., n)$, $\overline{T}_0$ is the average value of T_0. $\sigma_{W_k}^O$ is the standard deviation of $W_k\,(k = 1, 2, ..., n)$, $\sigma_{W_0}^O$ is the standard deviation of W_0 , $\overline{W}_k$ is the average value of $W_k\,(k = 1, 2, ..., n)$, $\overline{W}_0$ is the average value of W_0.

VI. A Numerical Example

In this section, we apply and evaluate the performance of the proposed method to address the problem of optimization of spare parts varieties. Firstly, we introduce the background on this simple system and the information of input and output factors. Then we calculate the optimal value of each DMU under SSOM. Finally, we provide some decision proposals on inventory management.

We focus on a depot which support eight airplanes in a mission. Ten items are selected randomly from the airplane parts list as an example to illustrate the proposed approach. The quantification results of qualitative indexes are obtained from expert elicitation as shown table II. The quantification results of deterministic indexes are determined by mission requirements and historical data, as shown in table III. The distributions of random variables are supposed to be normal distributions to reduce the computational complexity and the relative data are shown in table IV.

We regard each type of item as a DMU to calculate the optimal value in SSOM and then select the appropriate spare parts based on these optimal results. The input values and output values are taken from the quantification results as shown in table II to IV. Specifically, the belief degree α is 0.80, which means that the target DMU could meet the restrictions with the probability of 0.80. The optimal results solved by the SSOM is shown in table V.

According to the ranking criteria, the basic ranking order is DMU_4, DMU_5, DMU_6, DMU_7, DMU_8, DMU_9, DMU_{10}, DMU_1, DMU_2, DMU_3, in which DMU_4 to DMU_{10} are equally important. Based on the ranking order, the basic inventory policy is to store item 4 to item 10. Moreover, we could distinguish the equally important items by a single index. For example, cost is an important factor in the inventory management. Priority is usually given to low-cost items. Then the ranking order in terms of cost could be given as follows: DMU_9, DMU_{10}, DMU_7, DMU_6, DMU_8, DMU_4, DMU_5. The inventory policy could be adjusted based on the new ranking order when the budget is limited.

VII. Conclusions

We presented a stochastic DEA model for optimization under uncertainty, developed equivalent deterministic models to overcome difficulty in solving non-linear programming and then applied our approach to address the problem of optimizing the inventory policy of spare parts varieties. We proposed a comprehensive index system from the perspective of the product life-cycle process, which consisted of design factors, operation factors and support factors. Then we established a stochastic DEA model called SSOM considering the uncertain nature in parameters to select spare parts varieties with the constraints of multi-criteria. Some algorithms were employed to obtain the equivalent deterministic model of SSOM. In particular, the equivalent deterministic model of normal distributions was discussed. Finally, we applied our approach on a depot which serves eight airplanes for illustration and provided reasonable inventory management proposals based on the optimal results. We also provided a single criterion for equally important DMUs in terms of cost.

Decision makers can assign different values of belief degree α to different factors in terms of actual demand, which keeps the same for all random factors in the current model. Although the model in this paper deals with the optimization problem in spare parts varieties, it is a general model and can be applied in the other fields of multi-criteria optimization where random factors need to be considered. In actual implementation, decision makers can simplify the model as needed.

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TABLE II
QUANTIFICATION RESULTS OF QUALITATIVE INDEXES

Number	E	G	A	P
Item 1	7	7	3	3
Item 2	1	5	3	1
Item 3	9	7	3	1
Item 4	7	7	1	3
Item 5	5	9	1	3
Item 6	3	1	1	1
Item 7	7	5	1	3
Item 8	7	5	1	3
Item 9	7	7	3	3
Item 10	5	3	3	3

TABLE III
QUANTIFICATION RESULTS OF DETERMINISTIC INDEXES

Number	Z	N	S	L(day)	C(million)
Item 1	6	12	8	16	0.7
Item 2	6	4	8	8	0.9
Item 3	8	2	8	8	0.5
Item 4	2	1	8	8	3.7
Item 5	2	1	8	8	9.57
Item 6	5	3	8	7	0.7
Item 7	7	19	8	7	0.4
Item 8	15	25	8	7	1.2
Item 9	4	8	8	10	0.2
Item 10	6	4	8	10	0.3

TABLE IV
DISTRIBUTIONS OF RANDOM VARIABLES

Number	T(h)	F(h)	W(h)	M(h)	D(h)
Item 1	N(11, 0.78)	N(49103, 1054)	N(60724, 1067.6)	N(14.5, 2.77)	N(7.3, 0.83)
Item 2	N(7.9, 0.46)	N(134081 ,1659.5)	N(159089, 1089.7)	N(22.0, 3.02)	N(9.9, 2.02)
Item 3	N(0.5, 0.25)	N(78495,1347)	N(239800 , 3409.9)	N(17.6, 2.97)	N(3.7, 0.87)
Item 4	N(20, 0.38)	N(50466,1289)	N(678006 , 4590.6)	N(3.5 , 1.33)	N(11.4, 1.73)
Item 5	N(4.5, 1.32)	N(154077,2531.9)	N(149800 , 2063.2)	N(11.0, 0.80)	N(7.8, 0.96)
Item 6	N(2.7, 0.35)	N(70290, 1356.8)	N(132079, 1340.7)	N(2.5, 0.46)	N(12.1, 0.98)
Item 7	N(5.9, 2.32)	N(130839, 1567.5)	N(4765000, 3985.7)	N(10.5, 1.65)	N(15.6, 2.86)
Item 8	N(18.6, 5.22)	N(93675, 1029.6)	N(158900, 2183.4)	N(16.7, 2.77)	N(13.6, 2.18)
Item 9	N(1.9, 0.28)	N(165009, 1342.8)	N(367284, 3026.4)	N(17.9, 2.65)	N(8.7, 1.35)
Item 10	N(1.8, 0.27)	N(134200,1507.6)	N(529099, 4515.7)	N(4.5, 1)	N(5.3, 0.66)

TABLE V
OPTIMAL RESULTS

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8	DMU_9	DMU_{10}
θ	$6.23 * 10^4$	$1 * 10^5$	$4.95 * 10^5$	0	0	0	0	0	0	0

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Uncertainty theory as a basis for belief reliability

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ABSTRACT

Belief reliability is a newly developed, model-based reliability metric which considers both what we know (expressed as reliability models) and what we don't know (expressed as epistemic uncertainty in the reliability models) about the reliability. In this paper, we show that due to the explicit representation of epistemic uncertainty, belief reliability should not be regarded as a probability measure; rather, it should be treated as an uncertain measure in uncertainty theory. A minimal cut set-based method is developed to calculate the belief reliability of coherent systems. A numerical algorithm is, then, presented for belief reliability analysis based on fault tree models. The results of application show that the developed methods require less computations than the structure function-based method of classical reliability theory.

1. Introduction

Modern reliability engineering is increasingly looking at the model-based methods (cf. physics-of-failure (PoF) methods [5], structural reliability methods [6], etc.), where reliability is predicted exploiting deterministic failure behavior models whose parameter variations are assumed to be the only source of uncertainty [37]. In practice, however, apart from the random variations in the model parameters (often referred to as aleatory uncertainty [1]), the predicted reliability is also subject to the influence of epistemic uncertainty due to incomplete knowledge on the degradation and failure processes [20]: for example, the developed failure behavior model might not be able to accurately describe the actual failure process; besides, the precise values of the model parameters might not be accurately estimated [2,4], etc. In most existing model-based reliability assessment methods, however, the effect of epistemic uncertainty has not been considered.

Recently, a new metric of reliability, the belief reliability, has been defined to explicitly account for epistemic uncertainty in model-based reliability analysis and assessment [10,35,37]. The new reliability metric integrates the contributions of design margin, aleatory uncertainty and epistemic uncertainty and provides a more comprehensive and systematic description of reliability. Zeng et al. [37] presented a framework to evaluate the belief reliability where epistemic uncertainty is quantified by the effectiveness of the engineering analysis and assessment activities that contribute to the state of knowledge on the failure causes and processes. Belief reliability has been applied successfully on the reliability evaluation of hydraulic servo-actuators [35,37], DC regulated power supplies [10] and printed circuit boards [17], all of which are subject to the influence of epistemic uncertainty.

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Currently, the belief reliability of a component or a system can only be evaluated from its definition (i.e., based on design margin, aleatory uncertainty and epistemic uncertainty) [35]. In practice, we often need to calculate the belief reliability of a system based on the structure of the system and the belief reliabilities of its components (referred to as system reliability analysis in conventional reliability theories [31]). To address this problem, a mathematical theory should be determined as the mathematical foundation of belief reliability, based on which the system belief reliability analysis method can be developed. In literature, various mathematical theories have been used to describe epistemic uncertainty, e.g., probability theory (subjective interpretation [7]), evidence theory [29], possibility theory [8] and uncertainty theory [26], etc. Kang et al. [18] reviewed the theories and concluded that among them, uncertainty theory is the most suitable one for modeling belief reliability since it satisfies the Duality Axiom and adopts minimum operation as the Product Axiom, which are two essential requirements for a mathematical theory qualified to describe reliability under the influence of epistemic uncertainty. If either requirement is violated, misleading results might be reached when belief reliability is applied in practical applications (see Section 3.2 for a detailed discussion).

Uncertainty theory, proposed by Liu in 2007 [21] and refined by Liu in 2010 [24], is a branch of axiomatic mathematic founded on four axioms, the Normality, Duality, Subadditivity and Product Axiom. Currently, uncertainty theory has been widely applied in various fields, including portfolio selection [38], network science [14], option pricing [16], graph theory [13], transportation [32], supply chain [15], etc. The research of reliability in uncertainty theory started from [23], where Liu defined the reliability index and showed how to calculate the system reliability index from the system structure functions. In [27], the reliability indexes for redundant systems were calculated for the case in which the lifetimes of the components are uncertain variables. Zeng et al. [36] defined time-static and time-variant reliability in the context of uncertainty theory and developed calculation methods for the reliability indexes. Wen and Kang [30] developed an approach to calculate the reliability index when both uncertain variables and random variables are considered. Gao and Yao [12] investigated the importance index in the context of uncertainty theory. Age replacement and block replacement policies were also investigated with lifetimes described as uncertain variables [19,39,40].

Most existing system reliability analysis methods in uncertainty theory are based on structure functions (e.g., see [23,30]). Since they require enumerating all the possible combination of system states, the computational efficiency of the structure function-based methods are often unsatisfactory, especially for large and complex systems. In a previous study, minimal cut sets have been used to alleviate the computational burdens of the structure function-based methods [36]. However, the method developed in [36] requires independence among the minimal cut sets, which is a strong condition and restricts its application. In this paper, we show that the restriction is unnecessary and develop a minimal cut set-based method to calculate the belief reliability for a system with independent components.

The rest of this paper is organized as follows. Section 2 reviews the definition of belief reliability. In Section 3, we justify the choice of uncertainty theory as the mathematical foundation of belief reliability and give the definition of belief reliability in the context of uncertainty theory. Then, a system belief reliability analysis method is developed based on minimal cut sets in Section 4. In Section 5, a numerical algorithm is presented for belief reliability analysis based on fault tree models. The paper is concluded in Section 6 with discussions on possible future research directions.

2. Definition of belief reliability

In traditional model-based reliability methods, it is assumed that the failure behavior of a component or system is characterized by its performance margin$m$, which is modeled by:

$$m = g_m(\mathbf{x}), \tag{1}$$

where $m \leq 0$ indicates that the component or system fails and $m > 0$ indicates normal functioning; $g_m(\cdot)$ is developed by modeling the failure process [34]. Given the probability density functions of the input variables $\mathbf{x}$, denoted by $f_X(\mathbf{x})$, the reliability index can be calculated as

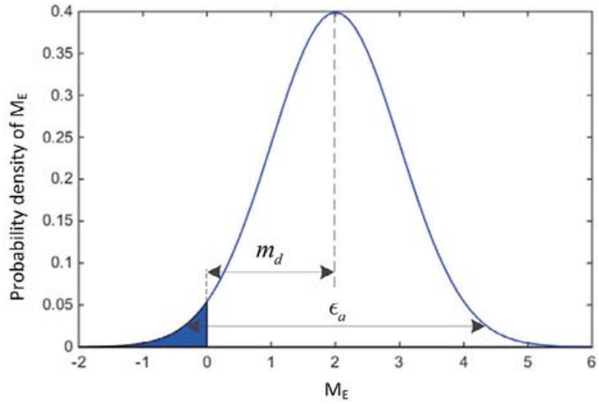
$$R_p = Pr(g_m(\mathbf{x}) > 0) = \int \cdots \int_{g_m(\mathbf{x})>0} f_X(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{2}$$

To differentiate it from belief reliability, the reliability index in (2) is referred to as probabilistic reliability in this paper.

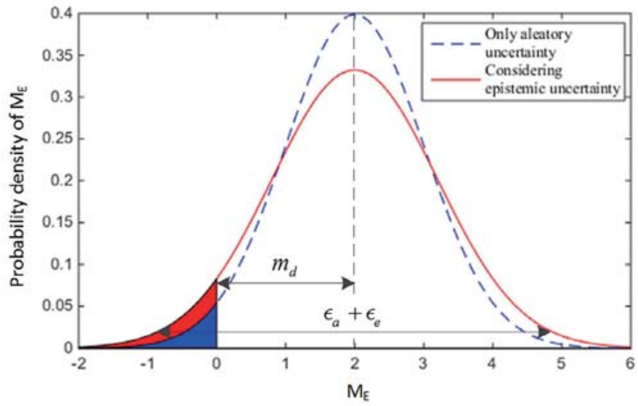
In the model-based reliability methods, a fundamental assumption is that, the reliability model is correct and accurate, so that all the uncertainty comes from the random variations in $\mathbf{x}$ (aleatory uncertainty). The validity of such an assumption heavily depends on the state-of-knowledge we have on the failure process. In a lot of practical applications, however, due to the limitation of the knowledge, the models in (1) and (2) might not be able to accurately capture the actual failure process. Besides, the precise values of the model parameters might not be accurately known to us. Therefore, the predicted reliability index is subject to an additional source of uncertainty, which arises from lack of knowledge and is referred to as epistemic uncertainty [41].

Belief reliability was proposed as a metric of reliability that explicitly accounts for epistemic uncertainty in reliability analysis and assessment [10,35,37]. Note that in (1) and (2), the probabilistic reliability R_p can be viewed as determined by deterministic designs and aleatory uncertainty in the design parameters. Deterministic designs are quantified by design margin m_d:

$$m_d = g_m(\mathbf{x}_N) \tag{3}$$



(a) Aleatory distribution of the performance margin



(b) Effect of epistemic uncertainty

Fig. 1. Epistemic uncertainty effect on the aleatory distribution of the performance margin (Adapted from [37]).

where $\mathbf{x}_N$ is the nominal values of the parameters. Aleatory uncertainty is measured by F_a, the factor of aleatory uncertainty, which is defined by:

$$F_a = \frac{m_d}{Z_{1-R_p}} \tag{4}$$

where R_p is given by (1) and (2); Z_α is the value of the inverse cumulative distribution function of a standard normal distribution evaluated at α. Let us define equivalent performance margin M_E as:

$$M_E = m_d + \epsilon_a, \tag{5}$$

where m_d is the design margin in (3) and $\epsilon_a \sim \text{Normal}(0, F_a^2)$ quantifies the effect of aleatory uncertainty. It is easy to verify that $M_E \sim \text{Normal}(m_d, F_a^2)$ and the probabilistic reliability R_p can be calculated as the probability that $M_E > 0$, as shown in Fig. 1 (a).

In belief reliability, epistemic uncertainty is described by introducing a factor of epistemic uncertainty, denoted by F_e, whose value is related to the state-of-knowledge of the failure processes and is measured based on the effectiveness of the engineering analysis and assessment activities for component and system reliability performance characterization [10,37]. An adjustment factor $\epsilon_e \sim \text{Normal}(0, F_e^2)$ is introduced to quantify the effect of epistemic uncertainty on the equivalent performance margin:

$$M_E = m_d + \epsilon_a + \epsilon_e. \tag{6}$$

Eq. (6) indicates that epistemic uncertainty introduces additional dispersion to the aleatory distribution of the equivalent performance margin, as shown in Fig. 1 (b). Considering (6) and the normality assumption on ϵ_a and ϵ_e, belief reliability is defined as:

Definition 1 (Belief reliability [37]). The reliability metric

$$R_B = \Phi_N\left(\frac{m_d}{\sqrt{F_a^2 + F_e^2}}\right) \tag{7}$$

is defined as belief reliability, where $\Phi_N(\cdot)$ is the cumulative distribution function of a standard normal random variable.

It can be shown from (7) that as $F_e \to 0$, $R_B \to R_p$, where R_p denotes the conventional model-based reliability metric calculated under the same conditions. This is natural, since $F_e \to 0$ indicates that there is no epistemic uncertainty and, therefore, the failure behavior can be accurately determined by the reliability models in (1) and (2).

In practical application, we always have $m_d > 0$ and $F_e \geq 0$ [37]. It is easy to verify from (7) that

$$R_B \leq R_p, \tag{8}$$

which shows that using belief reliability yields a more conservative evaluation result than using the probability-based reliability metric. The reason is that belief reliability considers the effect of insufficient knowledge on the estimated reliability, while the probability-based reliability metric implicitly assumes that knowledge is complete. It is the additional uncertainty caused by the insufficient knowledge that reduces our confidence on the reliability estimation.

3. Uncertainty theory as the mathematical foundation of belief reliability

In this section, we discuss the mathematical foundations of belief reliability and show that the new reliability metric should be modeled by uncertainty theory. Uncertainty theory is reviewed in Section 3.1. In Section 3.2, we explain the reasons to choose uncertainty theory as the mathematical foundation, and then define belief reliability as an uncertain measure.

3.1. Preliminaries of uncertainty theory

The first important concept in uncertainty theory is that of an event. Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event.

In uncertainty theory, the belief degree of an event is measured by its uncertain measure. An uncertain measure is a set function $\mathcal{M}$ from $\mathcal{L}$ to [0, 1] satisfying the following three axioms [21]:

Axiom 1 (Normality Axiom [21]). $\mathcal{M}\{\Gamma\} = 1$ for the univeral set Γ.

Axiom 2 (Duality Axiom [21]). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (Subadditivity Axiom [21]). For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$,

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}. \tag{9}$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space [21]. A product uncertain measure was defined by Liu [22] in order to obtain an uncertain measure of a compound event, thus producing the fourth axiom of uncertainty theory:

Axiom 4 (Product Axiom [22]). Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\} \tag{10}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e. , for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event [21].

In practice, an uncertain variable is described by the uncertainty distribution [21], defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \forall x \in \Re. \tag{11}$$

An uncertainty distribution is said to be regular if its inverse function $\Phi^{-1}(\cdot)$ exists and is unique for each $\alpha \in (0, 1)$ [24].

The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{m}\mathcal{M}\{\xi_i \in B_i\} \tag{12}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers [22].

Liu [24] developed operation laws for uncertain variables so that the distribution of functions of independent uncertain variables can be achieved. Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$, and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then, the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)\right). \tag{13}$$

3.2. Belief reliability as an uncertain measure

Belief reliability measures the degree to which we believe that a component or a system can perform its function as designed. In this subsection, we compare four mathematical theories commonly used to model belief degrees, probability theory (subjective interpretation [7]), evidence theory [29], possibility theory [8] and uncertainty theory [26], and choose among them the most appropriate one as the mathematical foundation for belief reliability.

In practice, how to calculate the belief degree of the intersection of events (more formally, the product event) is an important issue, since it is the basis of system reliability calculations. Based on how the belief degree of the intersection of events is calculated, the four theories can be divided into two groups. Probability theory and evidence theory comprise the first group, where the belief degree of the intersection of events is calculated by the product of the individual belief degrees (assuming independence among the individual events).

According to Liu [25], a premise of using the product operation to calculate the belief degree of the intersection of events is that the estimated belief degree for each individual event is close enough to the long-run cumulative frequency. As shown in (8), however, belief reliability is a more conservative reliability measure than the probabilistic reliability. If we use probability theory or evidence theory to model belief reliability, the conservatism in the component level will be distorted by the product operation, which might lead to counter-intuitive results when calculating system belief reliability. To illustrate this point, consider the following example.

Example 1. Consider a series system of 2000 components. Suppose for each component, $m_d = 9$ and $F_a = 0$. It is easy to verify that both the component and the system are unlikely to fail.

When using belief reliability as the reliability measure, we have to consider the effect of epistemic uncertainty, by evaluating our state of knowledge. Suppose for each component, we have $F_e = 3$. Then, from (7), the belief reliability of each component is $R_B = 0.9987$. If we regard belief reliability as a probability measure, the system belief reliability should be calculated by the product of the component belief reliabilities:

$$R_{B,S} = R_B^{2000} = 0.9987^{2000} = 0.074. \tag{14}$$

Based on the evaluation result in (14), the system is highly unreliable, which contradicts with our intuition.

Example 1 shows that to model belief reliability, we need a mathematical theory whose operation law of product events can compensate for the conservatism in the component-level belief reliability evaluation. Possibility theory provides an alternative solution by assuming that the product belief degree is the minimum one among all the individual events [8,33]. If we regard the component belief reliabilities in Example 1 as a possibility measure, according to [8], the system belief reliability is given by

$$R_{B,S} = \bigwedge_{i=1}^{2000} R_{B,i} = 0.9987, \tag{15}$$

which avoids the counter-intuitive result in Example 1. However, regarding belief reliability as a possibility measure introduces an issue: possibility measure does not follow the duality axiom, which might lead to other counter-intuitive results [24]. For instance, see Example 2.

Example 2. Assume that belief reliability R_B is a possibility measure. A possibility measure Π has the following properties [8]:

- $\Pi(\Omega) = 1$, where Ω is the universal set, and
- $\Pi(U \cup V) = \Pi(U) \vee \Pi(V)$, for any pair of disjoint sets U and V.

Since "working" and "failure" are two disjoint sets and their union is the universal set, from the above axioms, it is easy to show that for a given component or a system, either the reliability $R_B = 1$ or the unreliability $\overline{R_B} = 1$ which will confuse the decision maker when applied in practice.

From Examples 1 to 2, we can see that to model belief reliability, we need a mathematical theory which can compensate the conservatism in the individual belief degree and satisfy the duality axiom. Compared to probability theory, uncertainty theory differs in the Product Axiom, where a minimum operator is used instead of the product operator, indicating that the uncertainty theory is capable to compensate for the extra dispersion induced by epistemic uncertainty. Compared to possibility theory, uncertainty theory follows the Duality Axiom, which prevents the counter-intuitive examples such as that in Example 2. Hence, belief reliability is assumed to be an uncertain measure in this paper.

Definition 2 (Mathematical definition of belief reliability)**.** Let the universal set $\Gamma = \{\gamma_1, \gamma_2\}$, where γ_1 represents the working state of a system or component, while γ_2 represents the failure state. Then, belief reliability R_B is defined as the uncertain measure of the event $\Lambda_1 = \{\gamma_1\}$,

$$R_B = \mathcal{M}\{\Lambda_1\}. \tag{16}$$

Remark 1. From the Duality Axiom, we can calculate the belief unreliability:

$$\overline{R_B} = \mathcal{M}\{\Lambda_2\} = 1 - R_B, \tag{17}$$

which can also be seen from Fig. 1, since the areas of failure region and safe region sum up to 1.

4. Minimal cut set theorem

In this section, we show how to calculate the belief reliability of a coherent system by proving the Minimal Cut Set Theorem. Coherent system is the most widely applied system model in reliability theory, which describes the logic of binary monotone systems whose components are all relevant [3,28]. Commonly encountered examples of coherent systems include series systems, parallel systems, k-out-n:G systems, etc.

Let ξ_i, $1 \le i \le n$ and ξ denote the state of the ith component and of the system, respectively, where

$$\xi_i = \begin{cases} 1, \text{if the } i\text{th component is working}, \\ 0, \text{if the } i\text{th component fails}. \end{cases} \quad \xi = \begin{cases} 1, \text{if the system is working}, \\ 0, \text{if the system fails}. \end{cases} \tag{18}$$

The boolean variables ξ and ξ_i, $1 \le i \le n$ are referred to as state variables for the system and the components, respectively.

In coherent systems, ξ is a function of ξ_i, $1 \le i \le n$:

$$\xi = \phi(\mathbf{x}_\xi) = \phi(\xi_1, \xi_2, \cdots, \xi_n), \tag{19}$$

where $\mathbf{x}_\xi = [\xi_1, \xi_2, \cdots, \xi_n]$ is the state vector of the components. The function $\phi(\cdot)$ in (19) is the structure function of the coherent system.

The state variables ξ, ξ_i, $1 \le i \le n$ are all Boolean uncertain variables. Since ξ can be determined by ξ_1, ξ_2, $\cdots$, ξ_n via the structure function, ξ is a function of uncertain variables. Hence its uncertainty distribution can be obtained via the operation laws of uncertain variables [26]. Following the operation law for Boolean uncertain variables, Liu [23] proved the Reliability Index Theorem for coherent systems:

Theorem 1 (Reliability Index Theorem [23])**.** *Assume that a system contains uncertain elements ξ_1, ξ_2, $\cdots$, ξ_n and has a structure function ϕ. If ξ_1, ξ_2, $\cdots$, ξ_n are independent uncertain elements with reliability indices a_1, a_2, $\cdots$, a_n, respectively, then, the system reliability index a is*

$$a = \begin{cases} \sup\limits_{\phi(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if} \sup\limits_{\phi(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{\phi(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if} \sup\limits_{\phi(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{20}$$

where $x_i, i = 1, 2, \cdots, n$ take value either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0. \end{cases} \tag{21}$$

The proof of Theorem 1 can be found in [23].

Directly applying Theorem 1 to calculate belief reliability of a coherent system requires enumerating all possible combinations of ξ_i, which is tedious and hard to apply in practice. In order to simplify the evaluation processes, we develop a system belief reliability evaluation method for coherent systems based on the concept of minimal cut sets.

Definition 3 (Minimal cut set)**.** Suppose $\mathbf{x} = [x_1, x_2, \cdots, x_n]$ is the state vector of a coherent system whose structure function is ϕ. A vector $\mathbf{x}_a$ is called a minimal cut vector if $\phi(\mathbf{x}_a) = 0$ and $\phi(\mathbf{x}_b) = 1$, $\forall \mathbf{x}_b > \mathbf{x}_a$. By $\mathbf{x}_b > \mathbf{x}_a$, we mean $\mathbf{x}_{b,i} \ge \mathbf{x}_{a,i}$, $1 \le i \le n$ and there is at leat one i, $\mathbf{x}_{b,i} > x_{a,i}$.

Suppose x_C is a minimum cut vector. Let $C(x_C) = \{i : x_i = 0\}$. Then, $C(x_C)$ is referred to as a minimum cut set.

A minimal cut set is the smallest combination of components which will cause the systems failure if they all fail. In [36], the authors used minimal cut sets to reduce the computational costs in system belief reliability calculations. However, their method requires a strict assumption that all the minimal cut sets are independent. In this paper, we show that the restriction is unnecessary, by proving the Minimal Cut Set Theorem, which only requires independence among the components.

Theorem 2 (Minimal Cut Set Theorem)**.** *Consider a coherent system comprising n independent components with belief reliabilities $R_{B,i}, i = 1, 2, \ldots, n$. If the system contains m minimal cut sets, $C_1, C_2, \ldots, C_m$, then, the system belief reliability is*

$$R_{B,S} = \bigwedge_{1\le i\le m} \bigvee_{j\in C_i} R_{B,j}. \tag{22}$$

Proof. Without loss of generality, let us assume that the ith minimal cut set C_i contains n_i components. Let us also assume

$$\begin{aligned} &R_{B,11} \geq R_{B,12} \geq \cdots R_{B,1j} \geq \cdots \geq R_{B,1n_1}, \\ &R_{B,21} \geq R_{B,22} \geq \cdots R_{B,2j} \geq \cdots \geq R_{B,2n_2}, \\ &\qquad\vdots \\ &R_{B,m1} \geq R_{B,m2} \geq \cdots R_{B,mj} \geq \cdots \geq R_{B,mn_m}, \end{aligned}$$

and

$$R_{B,11} \geq R_{B,21} \geq \cdots R_{B,j1} \geq \cdots \geq R_{B,m1},$$

where $R_{B,\,ij}$ denotes the belief reliability of the jth component in the ith minimal cut set. In order to prove (22), we only have to prove

$$R_{B,S} = R_{B,m1}. \tag{23}$$

Eq. (23) comes from the fact that $R_{B,\,11}$, $R_{B,\,21}$, $\cdots$, $R_{B,\,m1}$ are the maximum component belief reliabilities for each minimal cut set, and $R_{B,\,m1}$ is the minimum among $R_{B,\,11}$, $R_{B,\,21}$, $\cdots$, $R_{B,\,m1}$.

The proof breaks into two cases:

1. If $R_{B,\,m1} < 0.5$:
Since $\phi(x_1, x_2, \cdots, x_n) = 1$ indicates that at least one component in each minimal cut set is working, it is easy to verify that

$$\sup_{\phi(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = \min_{1\leq i\leq m} \left\{ \max_{\phi_i(x_1,x_2,\cdots,x_{n_i})=1} \min_{1\leq j\leq n_i} \nu(x_{ij}) \right\} \tag{24}$$

where $\phi_i(x_1, x_2, \cdots, x_{n_i}) = \max_{1\leq j\leq n_i} x_{ij}$.
Since $R_{B,m1} \geq R_{B,m2} \geq \cdots R_{B,mj} \geq \cdots \geq R_{B,mn_m}$, we have

$$\begin{aligned} \max_{\phi_m(x_1,x_2,\cdots,x_{n_m})=1} \min_{1\leq j\leq n_m} \nu(x_{ij}) &= \min\left(R_{B,m1}, \min_{2\leq j\leq n_m} (1 - R_{B,mj})\right) \\ &= R_{B,m1}. \end{aligned} \tag{25}$$

For $1 \leq i \leq m-1$, if $R_{B,\,i1} \geq 0.5$, from Lemma 1 in Appendix A, we have

$$\max_{\phi_i(x_1,x_2,\cdots,x_{n_i})=1} \min_{1\leq j\leq n_i} \nu(x_{ij}) \geq 0.5 > R_{B,m1}; \tag{26}$$

if $R_{B,\,i1} < 0.5$, then, like (25), we can prove that

$$\max_{\phi_i(x_1,x_2,\cdots,x_{n_i})=1} \min_{1\leq j\leq n_i} \nu(x_{ij}) = R_{B,i1} \geq R_{B,m1}. \tag{27}$$

Substituting (26) and (27) into (24), we have

$$\sup_{\phi_i(x_1,x_2,\cdots,x_{n_i})=1} \min_{1\leq j\leq n_i} \nu(x_{ij}) = R_{B,m1} < 0.5. \tag{28}$$

Note that belief reliability is a reliability index. Then, from Theorem 1, $R_{B,S} = R_{B,m1}$.

2. If $R_{B,\,m1} \geq 0.5$:
Since $R_{B,\,11} \geq R_{B,\,21} \geq \cdots R_{B,\,j1} \geq \cdots \geq R_{B,\,m1} \geq 0.5$, from Lemma 1, we have

$$\sup_{\phi(x_1,x_2,\cdots,x_{n_i})=1} \min_{1\leq j\leq n_i} \nu_i(x_i) \geq 0.5. \tag{29}$$

Since $\phi_i(x_1, x_2, \cdots, x_{n_i}) = 0$ indicates that at least in one minimal cut set, all the components fail, we have

$$\begin{aligned} \sup_{\phi(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu(x_i) &= \max_{1\leq i\leq m} \min_{1\leq j\leq n_i} (1 - R_{B,ij}) \\ &= \max_{1\leq i\leq m} (1 - R_{B,i1}) = 1 - R_{B,m1}. \end{aligned} \tag{30}$$

Then, from Theorem 1,

$$R_{B,S} = 1 - \sup_{\phi(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu(x_i) = R_{B,m1}. \tag{31}$$

□

Example 3 (Belief reliability of a series system)**.** Consider a series system comprising n independent components with belief reliabilities $R_{B,i}, i = 1, 2, \ldots, n$. It is easy to show that the system has n minimal cut sets, $C_1 = \{1\}, C_2 = \{2\}, \ldots, C_n = \{n\}$. Therefore, from Theorem 2, the belief reliability of the system is

$$R_{B,S} = \bigwedge_{1 \le i \le n} R_{B,i}. \tag{32}$$

Ref. [23] also calculates the belief reliability of a series system using the Reliability Index Theorem. The result in (32) is the same as that from using Theorem 1 ([23]). However, using Theorem 1 requires $n \cdot 2^n$ comparisons, while using Theorem 2 requires only n comparisons. Therefore, the computational costs can be greatly reduced by using the Minimal Cut Set Theorem.

Example 4 (Belief reliability of a parallel system)**.** Consider a parallel system comprising n independent components with belief reliabilities $R_{B,i}, i = 1, 2, \ldots, n$. It is easy to show that the system has one minimal cut set, $C_1 = \{1, 2, \ldots, n\}$. Therefore, from Theorem 2, the system belief reliability is

$$R_{B,S} = \bigvee_{1 \le i \le n} R_{B,i}. \tag{33}$$

Ref. [23] also calculates the belief reliability of a parallel system using the Reliability Index Theorem. The result in (33) is the same as that from using Theorem 1 ([23]). However, using Theorem 1 requires $n \cdot 2^n$ comparisons, while using Theorem 2 requires only n comparisons. Therefore, the computational costs can be greatly reduced by using the Minimal Cut Set Theorem.

Example 5 (Belief reliability of a k-out-n:G system)**.** Consider a k-out-n:G system comprising n independent components with belief reliabilities $R_{B,i}, i = 1, 2, \ldots, n$. It is easy to show that the system has $C_n^{(k+1)}$ minimal cut sets. Each minimal cut set contains $k + 1$ components arbitrary chosen from the n components. Therefore, from Theorem 2, the belief reliability of the system is

$$R_{B,S} = R_{B,k}. \tag{34}$$

Ref. [23] also calculates the belief reliability of a k-out-n:G system using the Reliability Index Theorem. The result in (34) is the same as that from using Theorem 1 ([23]). However, using Theorem 1 requires $n \cdot 2^n$ comparisons, while using Theorem 2 requires only n comparisons. Therefore, the computational costs can be greatly reduced by using the Minimal Cut Set Theorem.

5. Fault tree analysis using belief reliability

In this section, we show how to calculate system belief reliability based on fault tree models. For this, we first show that Theorem 2 also applies to cut sets. A vector $\mathbf{x}_{CS}$ is a cut vector if $\phi(\mathbf{x}_{CS}) = 0$. Then, $CS = \{i : x_{CS,i} = 0\}$ is defined as a cut set. All minimal cut sets are cut sets; whereas, a cut set might be necessarily be a minimal cut set since it might contain redundant elements. If a cut set CS comprises of all the elements of a minimal cut set C and some redundant elements, C is said to be contained in CS.

Theorem 3 (Cut Set Theorem)**.** *Suppose that a coherent system has m minimal cut set CS_1, CS_2, $\cdots$, CS_m and $(l - m)$ cut sets $CS_{m+1}, CS_{m+2}, \cdots, CS_l$ that contain some minimal cut sets. Then, the system belief reliability can be calculated by*

$$R_{B,S} = \bigwedge_{1 \le i \le l} \bigvee_{j \in CS_i} R_{B,j}. \tag{35}$$

Proof. Let

$$R_{B,MCS} = \bigwedge_{1 \le i \le m} \bigvee_{j \in CS_i} R_{B,j}. \tag{36}$$

Without loss of generality, let us assume that CS_{m+1} contains CS_1 and belief reliabilities of the redundant components are $R_{B,R,1} \ge R_{B,R,2} \ge R_{B,R,n_R}$. Let $R_{B,\,1}$ denote the highest belief reliability among the components in CS_1.

If $R_{B,\,R,\,1} \le R_{B,\,1}$, immediately we have

$$R_{B,MCS} = \bigwedge_{1 \le i \le m+1} \bigvee_{j \in CS_i} R_{B,j}. \tag{37}$$

If $R_{B,\,R,\,1} > R_{B,\,1}$, (37) also holds since

$$\bigvee_{j \in CS_{m+1}} R_{B,j} = R_{B,R,1} > R_{B,1}. \tag{38}$$

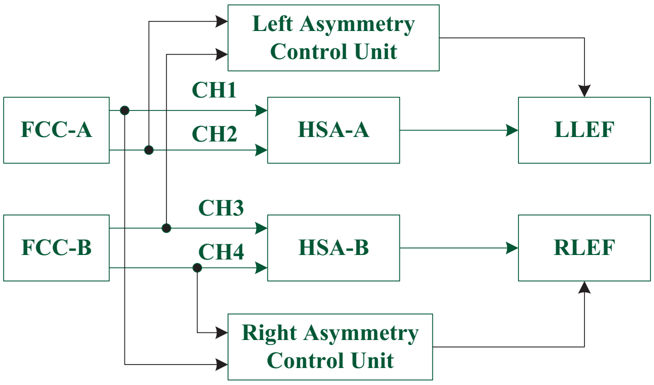


Fig. 2. Schematic diagram of the F-18 LLEF [9].

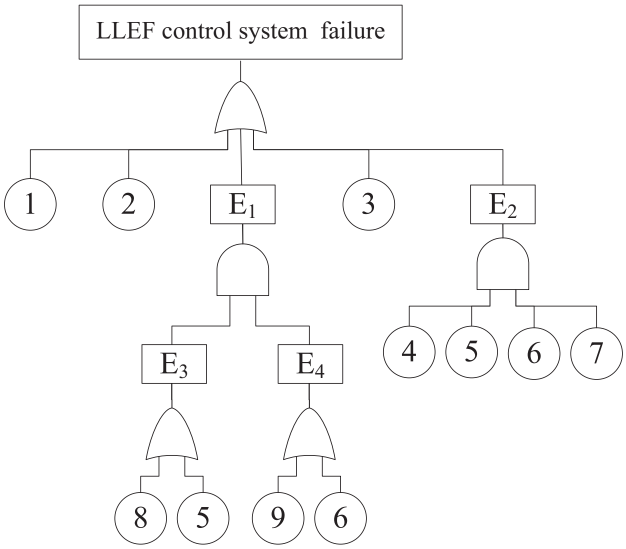


Fig. 3. The fault tree of the F-18 LLEF [9].

Similarly, we can prove that

$$\bigwedge_{1 \le i \le l} \bigvee_{j \in CS_i} R_{B,j} = R_{B,MCS}. \tag{39}$$

From Theorem 2, $R_{B,MCS} = R_{B,S}$. Hence, the theorem is proved. □

The cut sets required in (35) can be enumerated from the fault tree model using the MOCUS algorithm [11]. System belief reliability can, then, be calculated by Algorithm 1.

An engineering system, the left leading edge flap (LLEF) control subsystem of the F-18 air fighters [9], is used to demonstrate the developed system belief reliability analysis method. The schematic of the system is given in Fig. 2, where FCC represents flight control computer, CH represents channel, HSA represents hydraulic servo- actuator, LLEF represents left leading edge flap and RLEF represents right leading edge flap [9].

The failure behavior of the system can be described by a fault tree, as shown in Fig. 3 [9]. In Fig. 3, the basic events 1–9 represent the failure of HSA-A, left asymmetry control unit, LLEF, CH1, CH2, CH3, CH4, FCC-A and FCC-B, respectively.

The belief reliability of the components can be evaluated using the procedures in [37]. Suppose the component belief reliabilities are $R_{B,1} = 0.9688, R_{B,2} = 0.9200, R_{B,3} = 0.9500, R_{B,4} = 0.9000, R_{B,5} = 0.8000, R_{B,6} = 0.8800, R_{B,7} = 0.9600, R_{B,8} = 0.9700, R_{B,9} = 0.9500$, respectively. From Algorithm 1, the belief reliability of the system is

$$\begin{aligned} R_{B,S} \quad = \quad & R_{B,1} \wedge R_{B,2} \wedge R_{B,3} \wedge \\ & ((R_{B,5} \wedge R_{B,8}) \vee (R_{B,6} \wedge R_{B,9})) \wedge \\ & (R_{B,4} \vee R_{B,5} \vee R_{B,6} \vee R_{B,7}) \end{aligned} \tag{40}$$

Then, from (40), the belief reliability of the LLEF control system is $R_{B,S} = R_{B,6} = 0.8800$.

Algorithm 1 Belief reliability analysis based on fault tree.

1: Do a depth-first-search for the logic gates in the fault tree.
2: For each logic gate, calculate the belief reliability for its output event:

$$R_{B,out} = \begin{cases} \bigwedge_{1\le i\le n} R_{B,in,i}, & \text{for an OR gate,} \\ \bigvee_{1\le i\le n} R_{B,in,i}, & \text{for an AND gate,} \end{cases} \tag{39}$$

3: $R_{B,S} \leftarrow R_{B,out,TE}$, where TE represents top event.
4: **return** $R_{B,S}$.

The structure function-based method is also used to evaluate the system belief reliability. To do this, all the possible combinations of the system states need to be enumerated, which, in this case, are $2^9 = 512$ states. Then, the system belief reliability is calculated based on (20). The calculated system belief reliability is $R_{B,S} = 0.8800$, which is the same as the one from Algorithm 1. According to (20), the structure function-based method requires $n \times 2^n = 4608$ comparisons, where n is the number of components. Algorithm 1, however, requires only 10 comparisons according to (40). The results demonstrate that using the developed methods can help to improve the computational efficiency of system belief reliability analysis.

6. Conclusion

In this paper, belief reliability was defined as an uncertain measure in uncertainty theory, due to the explicit representation of epistemic uncertainty. The Minimal Cut Set Theorem was proved, which shows how to calculate the belief reliability for coherent systems based on minimal cut sets. A system belief reliability analysis method is, then, developed based on fault tree models and applied on some numerical case studies. A comparison to the existing structure function-based method shows that the developed methods reduces the computational costs in system belief reliability analysis.

In this paper, we only consider binary systems. Many practical systems, however, are multi-state. In the future, the belief reliability evaluation method will be extended to multi-state system models. Also, the belief reliability considered in this paper is independent of time. How to model the time-dependent belief reliability is another future research direction. Besides, it should be noted that the belief reliability analysis method developed in this paper assumes that the system reliability model is accurate. Epistemic uncertainty only exists in the component level. In practice, however, epistemic uncertainty might also affect system reliability model and should be considered in future researches.

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Appendix A. Lemma 1 and its proof

Lemma 1. *Consider a coherent system comprising n independent components with belief reliabilities* $R_{B,i}, i = 1, 2, \cdots, n$, *where* $R_{B,\,1} \ge R_{B,\,2} \ge \cdots \ge R_{B,\,n}$. *If the structure function of the system* ϕ *is:*

$$\phi(x_1, x_2, \cdots, x_n) = \max_{1\le i\le n} x_i, \tag{A.1}$$

and there is at least one $R_{B,\,i}$ *such that* $R_{B,\,i} \ge 0.5$, *then we have*

$$\sup_{\phi(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) \ge 0.5. \tag{A.2}$$

Proof. The proof breaks into two cases:

1. If $R_{B,\,n} \ge 0.5$:
   Since $\phi(1, 1, \cdots, 1) = 1$, we have

$$\sup_{\phi(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) \ge \min_{1\le i\le n} \nu_i(1) = R_{B,n} \ge 0.5. \tag{A.3}$$

2. If $R_n < 0.5$:

Without loss of generality, we assume that there exists a $k, k \in [1, n-1]$, such that $R_{B,\,k} \geq 0.5$. Since $R_n < 0.5$, there exists a $j \in (k, n)$, where $R_j \geq 0.5 \geq R_{j+1}$. It is easy to verify that $\phi(x_1, x_2, \cdots, x_n) = 1$ where

$$x_i = \begin{cases} 1, i = 1, 2, \cdots, j \\ 0, i = j, j+1, \cdots, n. \end{cases} \tag{A.4}$$

Besides, for the x_i, $1 \leq i \leq n$ in (A.4), we have

$$\min_{1\leq i\leq n} \nu_i(x_i) = \min\left(\min_{1\leq i\leq j} \nu_j(1), \min_{j+1\leq i\leq n} \nu_j(0)\right) \geq 0.5. \tag{A.5}$$

Therefore,

$$\sup_{\phi(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) \geq 0.5. \tag{A.6}$$

□

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Belief Reliability Distribution Based on Maximum Entropy Principle

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ABSTRACT Belief reliability is a new reliability metric based on the uncertainty theory, which aims to measure system performance incorporating the influences from design margin, aleatory uncertainty, and epistemic uncertainty. A key point in belief reliability is to determine the belief reliability distribution based on the actual conditions, which, however, could be difficult when available information is limited. This paper proposes an optimal model to determine the belief reliability distribution based on the maximum entropy principle when kth moments of what can be obtained. An estimation method using linear interpolation and a genetic algorithm is subsequently applied to the optimal model. When only the expected value and the variance are available, the optimal results are in accordance with the maximum entropy principle. It could be observed in the sensitivity analysis that the accuracy of the optimal results is a decreasing function of the width of variances and an increasing function of the number of interpolation points. Therefore, researchers could adapt to different widths of variances and requirements of accuracy by adjusting the number of interpolation points. It could be concluded that this new method to acquire belief reliability distribution is important in the application of belief reliability.

INDEX TERMS Belief reliability distribution, maximum entropy principle, uncertain variable, uncertain distribution.

I. INTRODUCTION

With urgent requirements for the accuracy of the products reliability assessment, the treatment of uncertainties has attracted much attention. Generally, uncertainties can be classified into two types, aleatory uncertainty and epistemic uncertainty. Aleatory uncertainty describes the uncertainty inherent in the physical behavior of the system, and epistemic uncertainty is attributable to the lack of data and information. Probabilistic method can successfully deal with the aleatory uncertainty however it has obvious drawbacks on the treatment of epistemic uncertainty. In 2007, Liu [1] founded uncertainty theory to deal with human's subjective uncertainty by belief degree mathematically and in 2010, Liu [2] perfected it based on normality, duality, subadditivity and product axioms. Based on uncertainty theory, Zeng *et al.* [3] defined belief reliability as the uncertainty measure of the system to perform specific functions within given time under given operating conditions. Zeng *et al.* [4] developed an evaluation method for component belief reliability, which incorporates the impacts from design margin, aleatory uncertainty and epistemic uncertainty. The issue of quantifying the effect from epistemic uncertainty is addressed using a method, which is established based on the performance of engineering activities related to reduce epistemic uncertainties [5], [6]. However, it is still challenging to widely employ belief reliability in reliability engineering due to the scant methods to acquire belief reliability distributions.

Belief reliability distribution is inherently the uncertainty distribution applied in belief reliability. Researchers have explored several methods to get uncertainty distributions. Liu [2] designed uncertain statistics as a methodology for collecting and interpreting experts' experimental data by uncertainty theory and then proposed a questionnaire survey for collecting expert's experimental data. Chen and Ralescu [7] employed uncertain statistics to estimate the travel distance between Beijing and Tianjin and proposed B-spline interpolation to fit a continuous uncertainty distribution.

Gao and Yao [8] designed a procedure of the Delphi method for determining the uncertainty distribution. Both the B-spline interpolation and the Delphi method can adapt to the cases where uncertainty distributions are unknown. When the form of an uncertainty distribution is certain, Liu [2] utilized the principle of the least squares to estimate the parameters of the uncertainty distribution and Wang and Peng [9] proposed a method of moments for calculating the unknown parameters of the uncertainty distribution.

However, in practice, only partial information about an uncertain variable is available and there are infinite numbers of uncertainty distributions that are in accordance with the given information. In such cases, the existing methods cannot determine its uncertainty distribution.

The entropy is a measurement of the degree of uncertainty. For random cases, Jaynes [10] suggested choosing the distribution which has the maximum entropy. In uncertainty theory, Liu [11] proposed the definition of uncertainty entropy resulting from information deficiency to provide a quantitative measurement for the degree of uncertainty of uncertain variables. Chen and Dai [12] proved the maximum entropy principle when the expected value and the variance are finite. This paper will investigate the maximum entropy method and propose an optimal model to estimate belief reliability distribution based on the maximum entropy principle when k-th moments can be obtained.

The paper is structured as follows. Some basic concepts on uncertainty theory will be introduced in Section 2. Subsequently, basic definitions on belief reliability and belief reliability distribution will be provided and a model based on the maximum entropy principle will be proposed to estimate belief reliability distribution in Section 3. The estimation to the proposed model will be discussed with linear interpolation and genetic algorithm (GA) in Section 4. The proposed model will be verified in Section 5 and a sensitivity analysis will be conducted on the number of interpolation points and the width of variances in the same section. The conclusions on belief reliability distribution based on the maximum entropy principle will be discussed in Section 6.

II. PRELIMINARIES

Uncertainty theory was founded by Liu [1] in 2007 and refined by Liu [2] in 2010. Following that, uncertain process [13], uncertain differential equations [13], uncertain calculus [11] and uncertain programming [14] were proposed. Uncertainty theory has been successfully applied in various areas, including finance [15], reliability [8] and graph [16]. Some basic concepts in uncertainty theory will be stated in this section.

Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Each element Λ in Γ is called an event. Liu [1] defined an uncertain measure by the following axioms:

Axiom 1 (Normality Axiom): $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (Duality Axiom): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^C\} = 1$ for any event Λ.

Axiom 3 (Subadditivity Axiom): For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}, \tag{1}$$

where $\bigcup_{i=1}^{\infty}\Lambda_i$ is the union of Λ_i, $i = 1, 2, \cdots$.

Furthermore, Liu [11] defined a product uncertain measure by the fourth axiom:

$$\mathcal{M}\left\{\prod_{i=1}^{\infty}\Lambda_i\right\} = \bigwedge_{i=1}^{\infty}\mathcal{M}_i\{\Lambda_i\} \tag{2}$$

where $\mathcal{L}_i$ are σ-algebras over Γ_i, Λ_i are arbitrarily chosen events from $\mathcal{L}_i$ for $i = 1, 2, \cdots$, respectively, and $\prod_{i=1}^{\infty}\Lambda_i$ is the intersection of Λ_i, $i = 1, 2, \cdots$.

Definition 1 (See Liu [1]): Let Γ be a nonempty set, let $\mathcal{L}$ be a σ-algebra over Γ, and let $\mathcal{M}$ be an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2 (See Liu [1]): An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, γ is the element in Γ, i.e., for any Borel set $\mathcal{B}$ of real numbers, we have

$$\{\xi \in \mathcal{B}\} = \{\gamma \in \Gamma | g(\gamma) \in \mathcal{B}\} \in \mathcal{L}. \tag{3}$$

Definition 3 (See Liu [1]): The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\} \tag{4}$$

for any real number x.

Example 1: An uncertain variable ξ is called normal variable if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathcal{R} \tag{5}$$

denoted by $\mathcal{N}(\mu, \sigma)$ where μ and σ are real numbers with $\sigma > 0$.

Definition 4 (See Liu [1]): Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\cdot)$ is called the inverse uncertainty distribution of ξ.

Example 2: The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(\mu, \sigma)$ is

$$\Phi_x^{-1}(\alpha) = \mu + \frac{\sigma\sqrt{3}}{\pi} ln\frac{\alpha}{1-\alpha}, \tag{6}$$

where α is the belief degree.

III. BELIEF RELIABILITY AND ITS DISTRIBUTION MODEL

A. BASIC DEFINITIONS AND EXAMPLES

Definition 5 (Belief Reliability): Let a product state variable ξ be an uncertain variable, and Ξ be the feasible domain of a product state. Then the belief reliability is defined as the uncertain measure that the product state is within the feasible domain, i.e.,

$$R_B = \mathcal{M}\{\xi \in \Xi\}. \tag{7}$$

In ***Definition 5***, the state variable ξ describes the product's behavior, while the feasible domain Ξ is a reflection of failure criteria. In reliability engineering, since the product's behavior and the failure criteria usually vary with time [17], both ξ and Ξ can be relevant to time t. In this case, the belief reliability metric will be a function of t, denoted by $R_B(t)$.

Example 3: The state variable ξ can represent the product failure time T which describes system failure behaviors. The product is regarded be reliable at time t if the failure time is larger than t. Thus, the belief reliability of the product at time t can be obtained by letting $\Xi = [t, +\infty)$, i.e. Ξ is relevant to t and $R_B(t)$ can be calculated by

$$R_B(t) = \mathcal{M}\{T > t\}. \tag{8}$$

Example 4: The state variable ξ can also represent the performance margin m of a product, which describes system operation behaviors. m describes the distance between a performance parameter and the associated failure threshold. Therefore, Ξ should be $(0, +\infty)$ and the belief reliability of the product can be written as

$$R_B = \mathcal{M}\{m > 0\}. \tag{9}$$

If we consider the degradation process of the performance margin, i.e., ξ is relevant to t, $R_B(t)$ will be

$$R_B(t) = \mathcal{M}\{m(t) > 0\}. \tag{10}$$

Definition 6 (Belief Reliability Distribution): Assume that a product state variable ξ is an uncertain variable, then the uncertainty distribution of ξ is defined as ***Belief Reliability Distribution***.

Example 5: When the state variable ξ represents the product failure time T. Then the uncertainty distribution Φ of T is belief reliability distribution.

Example 6: When the state variable ξ represents the product performance margin m. Then belief reliability distribution Ψ will be the uncertain measure of m, denoted as

$$\Psi(x) = \mathcal{M}\{m \le x\}. \tag{11}$$

B. BELIEF RELIABILITY DISTRIBUTION MODEL

The entropy measures the degree of uncertainty while uncertainty entropy serves as a quantitative measurement of the degree of uncertainty of uncertain variables. When only partial information is accessible, such as k-th moments, there are infinite numbers of uncertainty distributions that are consistent with the provided information. Here we employ the maximum entropy principle to ascertain the belief reliability distribution.

Relative symbols and notations are introduced briefly as follows:

ξ: an uncertain variable,

$\Phi(x)$: an uncertainty distribution of ξ,

μ_k: the k-th moment of uncertain variable ξ, $k = 1, 2, 3, \cdots$.

Definition 7 (See Liu [11]): Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is determined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))dx \tag{12}$$

where $S(t) = -tlnt - (1-t)ln(1-t)$.

Definition 8: (See Liu and Chen [18]): Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k d\Phi(x). \tag{13}$$

The optimal model is written as:

$$\begin{cases} \max H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))dx \\ s.t. \int_{-\infty}^{+\infty} x^k d\Phi(x) = \mu_k, \quad \text{for } k = 1, 2, 3, \cdots \end{cases} \tag{14}$$

More specifically,

$$\begin{cases} \max H[\xi] = -\int_{-\infty}^{+\infty} \Phi(x)ln(\Phi(x)) \\ \qquad\qquad + (1-\Phi(x))ln(1-\Phi(x))dx \\ s.t. \int_{-\infty}^{+\infty} x^k d\Phi(x) = \mu_k, \quad \text{for } k = 1, 2, 3, \cdots \end{cases} \tag{15}$$

IV. ESTIMATION TO BELIEF RELIABILITY DISTRIBUTION MODEL

This section discusses the estimation to the optimal model, which approximates belief reliability distribution based on the maximum entropy principle. Since the form of the belief reliability distribution is unknown, it is intuitive to apply the discretization method to obtain the approximate solution of the distribution. To obtain these discrete data, GA method is adopted to find the global optimum solution with the constraints on k-th moments. Subsequently, the linear interpolation method is used in this paper to estimate the belief reliability distribution.

Belief reliability distribution is discretized into the form of a piecewise linear function as shown in Eq.(16).

$$\Phi(x) = \begin{cases} 0 & \text{if } x < x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i} & \text{if } x_i < x < x_{i+1} \\ 1 & \text{if } x > x_N. \end{cases} \tag{16}$$

Then the Eq.(13) can be written as follows:

$$\mu_k = E[\xi^k] = \sum_{i=1}^{N-1} \frac{(\alpha_{i+1} - \alpha_i)(x_{i+1}^{k+1} - x_i^{k+1})}{(k+1)(x_{i+1} - x_i)}, \quad k = 1, 2, \cdots. \tag{17}$$

When the belief degrees satisfy:

$$\alpha(i) = (2i - 1)/2N, \quad i = 1, 2, \cdots, N \tag{18}$$

Eq.(12) can be written as:

$$H = N \times \sum_{i=1}^{N-1} p(i) \times (x_{i+1} - x_i) \tag{19}$$

where $p(i)$ is calculated by the Eq.(20).

$$\begin{aligned} p(i) = & -0.5(\alpha(i+1)^2 ln(\alpha(i+1)) - \alpha(i)^2 ln(\alpha(i))) \\ & + 0.25(\alpha(i+1)^2 - \alpha(i)^2) \\ & - 0.5((1-\alpha(i))^2 ln((1-\alpha(i))) \\ & - (1-\alpha(i+1))^2 ln((1-\alpha(i+1)))) \\ & + 0.25((1-\alpha(i))^2 - (1-\alpha(i))^2) \end{aligned} \tag{20}$$

Thus, the optimal model (15) is a non-linear programming problem as follows:

$$\begin{cases} \max H[\xi] = N \times \sum_{i=1}^{N-1} p(i) \times (x_{i+1} - x_i) \\ s.t. \sum_{i=1}^{N-1} \frac{(\alpha_{i+1} - \alpha_i)(x_{i+1}^{k+1} - x_i^{k+1})}{(k+1)(x_{i+1} - x_i)} - \mu_k = 0, \\ \qquad\qquad k = 1, 2, \cdots \\ x_1 < x_2 < \cdots < x_N \end{cases} \tag{21}$$

where $p(i)$ is calculated by Eq.(20).

Then genetic algorithm is applied to solve this non-linear programming problem. Finally, the approximation to the belief reliability distribution can be obtained by linear interpolation methods.

In summary, the estimation to the belief reliability distribution model can be divided into five steps, which is concisely illustrated in the Fig.1.

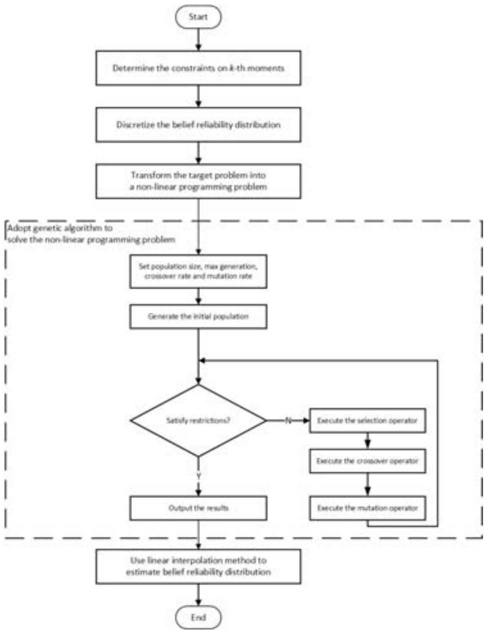

FIGURE 1. Flow chart of the estimation to the belief reliability distribution model.

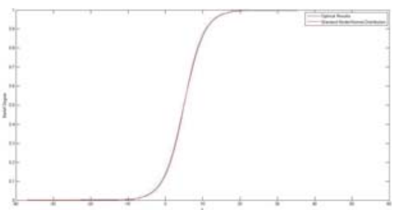

FIGURE 2. Optimal results and the standard model at $\mu = 5, \sigma^2 = 25, N = 500$.

V. MODEL VERIFICATION AND SENSITIVITY ANALYSIS

This section will briefly introduce the maximum entropy principle proved by Chen and Dai [12] and then verify the proposed optimal model based on this principle. Moreover, a sensitivity analysis is conducted on the effect of the width of variances and the number of interpolation points.

A. MODEL VERIFICATION

In uncertainty theory, Chen and Dai [12] proved a theorem called maximum entropy principle when the expected value and variance are known.

Theorem 1 (Maximum Entropy Principle): Let ξ be an uncertain variable with finite expected value μ and variance σ^2. Then

$$H[\xi] \leq \frac{\pi\sigma}{\sqrt{3}} \tag{22}$$

and the equality holds if is a normal uncertain variable with expected value e and variance σ^2 , i.e. $\mathcal{N}(\mu, \sigma)$.

According to the maximum entropy principle theorem, belief reliability distributions based on maximum entropy principle are determined when the 1st and 2nd moments are known.

When the expected value $\mu = 5$, the variance $\sigma^2 = 25$ and the number of interpolation points $N = 500$, the optimal results and the standard model are shown in Fig.2. The red line represents the standard model, normal uncertainty distribution $\mathcal{N}(5, 5)$, and the blue one shows the estimation to the proposed optimal model. As demonstrated in the Fig.2, there is not a great difference between the optimal results and the ideal results, which leads to the conclusion that the estimation by using linear interpolation and GA is effective to the optimal model. Fig.3 shows that absolute errors between the optimal results and standard model. As shown in Fig.3, the absolute errors are no more than 0.015, which also shows there is not a great difference between the optimal results and the ideal results. It could be concluded from the Fig.2 and Fig.3 that the optimal results are consistent with the standard model and the proposed estimating approach is effective to determine belief reliability distribution based on the maximum entropy principle.

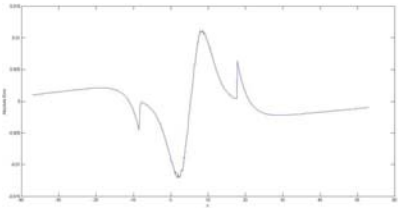

FIGURE 3. Absolute errors between optimal results and standard model at $\mu = 5, \sigma^2 = 25, N = 500$.

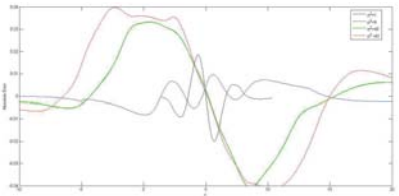

FIGURE 4. Absolute errors between optimal results and standard model at $\mu = 5, N = 500$.

B. SENSITIVITY ANALYSIS

As illustrated in Eq.(22), uncertainty entropy H is associated with variances. A sensitivity analysis on the width of variances is conducted to investigate the relationship between the width of variances and the optimal results.

In Fig.4, the four curves show the fluctuation of the absolute errors between optimal results and standard model at $\mu = 5, N = 500$ with variances at 1,9,49,81, respectively. As can be seen from the figure, there is an increasing tendency of the degree of the fluctuations with the width of variances raising. In other words, it could be implied that the accuracy of the optimal results decreases as the variances increase when the expected value and the number of interpolation points keep the same.

Moreover, the number of interpolation points N also has a significant influence on the optimal results. A sensitivity analysis on the number of interpolation points is conducted to explore the connection between the number of interpolation points and the optimal results.

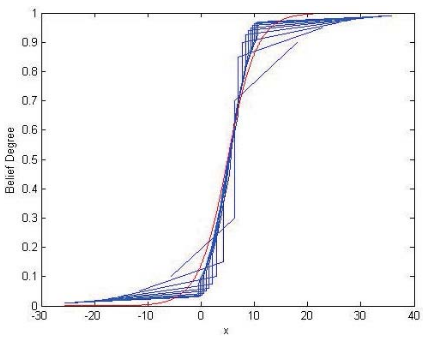

FIGURE 5. Optimal results and the standard model at at $\mu = 5, \sigma^2 = 25, N = 5:5:50$.

Fig.5 shows the number of interpolation points from 5 to 50 with the step length 5 when the expected value $\mu = 5$ and the variance $\sigma^2 = 25$ and Fig.6 shows the number of interpolation points from 55 to 100 with the step length 5 when the expected value and the variance keep the same. As shown in Fig.5, the fitting results of linear part are growing better as the number of interpolation is increasing. As shown in Fig.6, the fitting results of non-linear part are approaching the standard model as the number of interpolation points is growing. It could be inferred that the larger the number of interpolation points is, the better the optimal results are.

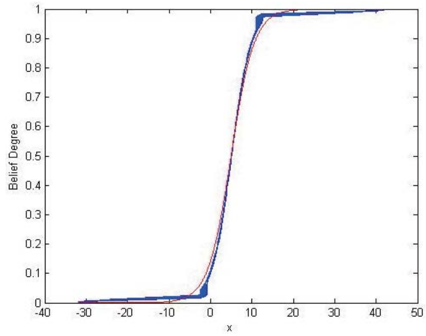

FIGURE 6. Optimal results and the standard model at $\mu = 5, \sigma^2 = 25, N = 55:5:100$.

From the above analysis, it could be observed that the accuracy of the optimal results is a decreasing function of the width of variances and an increasing function of the number of interpolation points.

VI. DISCUSSIONS AND CONCLUSIONS

This paper specified the definition of belief reliability and belief reliability distribution, extended the application of maximum entropy principle in uncertainty theory, and proposed an optimal model based on maximum entropy principle to estimate belief reliability distribution and an approach to estimate the optimal model using linear interpolation and genetic algorithm. According to the theorem proved by Chen and Dai [12], the proposed estimating method is effective to determine the belief reliability distribution. The estimating results are sensitive to the width of the variances and the number of interpolation points. Based on the results of the sensitivity analysis, when the number of interpolation points keeps still, the accuracy of the optimal results decreases as the width of the variance increases. In addition, the accuracy of the optimal results is an increasing function of the number of interpolation points when the width of the variance keeps the same. In actual situations, it is possible to obtain more accurate optimal results when we increase the number of interpolation points. Besides, the number of interpolation points also reflects the data density of belief degree according to Eq. (18). Therefore, when we only concentrate on belief degree around 0.5, we could adopt a small number of interpolation points to get satisfying optimal results. By contrast, when we focus on belief degree near to 0 or 1, we have to adopt a large number of interpolation points to obtain reasonable optimal results.

The proposed optimal model to estimate belief reliability distribution based on the maximum entropy principle can be applicable to the cases when k-th moments of an uncertain state variable are available, which is important in the development of belief reliability. Moreover, the proposed approach also provides a new approach to obtain uncertainty distributions in uncertainty theory.

The estimating approach applied is a simple but time-consuming one to obtain optimal results. Therefore, the alternative of the estimation deserves further investigation. Moreover, this paper only considers k-th moments as the constraints of the optimal model. More information could be included in the optimal model to adapt to diverse cases.

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Accelerated degradation model based on geometric Liu process

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ABSTRACT: Through evaluating stress levels, accelerated degradation testing (ADT) can obtain degradation data in a limited period of time and then use these data for reliability evaluations. However, because of the high price of the test items or the test equipment, the sample size used in ADT is usually small, which causes a lack of knowledge on recognizing the population and then lead to the epistemic uncertainty. The small sample problem makes the probability theory based models, which need large samples, not appropriate any more. To address this problem, based on the uncertain theory, this paper uses the general geometric Liu process to construct an uncertain acceleration degradation model, and gives the corresponding statistical analysis method with objective measures. A carbon-film resistors case is used to illustrate the proposed methodology, and discussions are conducted on the sensitivity analysis of the proposed methodology to the sample sizes. Results show that the proposed methodology is a suitable choice for the reliability evaluations of ADT data under small sample situations.

1 INTRODUCTION

Products' reliability and lifetime are usually assessed by the life testing that uses time-to-failure data. But for highly reliable products, there are usually few or even no failure during the life testing, which makes it unappropriated to use life testing to assess these products' reliability and lifetime. Therefore, the accelerated reliability testing (ADT) has attracted much attention and been widely applied. ADT can obtain degradation data in a limited period of time by evaluating stress levels, and then uses these data for the reliability and lifetime assessments.

In a standard ADT analysis, there is a degradation model and an acceleration model. The degradation model describes the degradation paths under each stress level. Some of the parameters are assumed to be functions of stress levels, i.e., the acceleration model. In general, there are two broad categories of degradation models based on the probability theory, which are the degradation path models (Meeker et al., 1998) and the stochastic process models (Ye and Xie, 2015). These models are suitable for the situation where there are large samples. But in practical applications, the sample size in ADT is usually small due to the high price of the test items or the test equipment, which will cause a lack of knowledge on recognizing the population, and then lead to the epistemic uncertainty. Therefore, the probability theory based models are not appropriate for the small sample situation.

To quantify the epistemic uncertainty, various methods have been applied by utilizing subjective information such as belief degrees, including the Bayesian method (Li and Meeker, 2014), the interval analysis (Moore et al., 2009), and the fuzzy probability theory (Beer et al., 2013) . The prior distributions, the intervals or the fuzzy variables are used respectively in these methods to utilize subjective information to quantify the epistemic uncertainty (Kang et al., 2016).

However, there still exit some problems. On the one hand, these methods quantify the epistemic uncertainty by subjective measures, which could result in different results from different researchers. On the other hand, these methods origin from the probability theory, which makes them unsuitable for the ADT data with small sample size.

Motivated by these problems, the uncertainty theory proposed by Liu (Liu, 2015) is introduced to the field of ADT modeling. The uncertainty theory is a branch of mathematics for modeling belief degrees and is used for the small sample (or even no sample) situations (Liu, 2012). It has been widely used in many fields such as risk assessment (Liu, 2010), reliability analysis (Zeng et al., 2013), supply chain (Huang et al., 2016), and so on.

In this paper, based on the uncertainty theory, a positive uncertain process named the general geometric Liu process is proposed to construct an uncertain accelerated degradation model, and the statistical analysis method for parameter estimations is proposed correspondingly. The proposed methodology quantifies the epistemic uncertainty by objective measures. The rest of the paper is organized as follows. Section 2 introduces preliminaries about the uncertainty theory. Section 3 presents the uncertain accelerated degradation model, derives the reliability and lifetime distributions and gives the corresponding statistical analysis method. Section 4 conducts the case study and the sensitivity analysis. Section 5 concludes the paper.

2 PRELIMINARIES

In this section, we introduce some preliminaries about uncertain measure, uncertain variable, and uncertain process that will be used in the subsequent sections.

Definition 1 (Liu, 2015): Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called a measurable set. A set function $\mathcal{M}$ from $\mathcal{L}$ to [0, 1] is called an uncertain measure if it satisfies the following axioms:

1) Normality axiom: $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

2) Duality axiom: $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

3) Subadditivity axiom: For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$ we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\left\{\Lambda_i\right\}. \tag{1}$$

4) Product axiom (Liu, 2009): Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for k=1, 2, … , then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\left\{\Lambda_k\right\}, \tag{2}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for k=1,2,…, respectively.

Definition 2 (Liu, 2015): Liu introduced the concept of uncertainty distribution to describe uncertain variables. The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \quad \forall x \in \mathcal{R}. \tag{3}$$

Let be ξ an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Definition 3 (Liu, 2008): Let T be a totally ordered set (that is usually "time"), and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is defined as a measurable function from $T\times(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set

$$\{X_t \in B\} = \{\gamma \in \Gamma \mid X_t(\gamma) \in \mathrm{B}\}\} \tag{4}$$

is an event. In other words, an uncertain process is a sequence of uncertain variables indexed by time.

Definition 4 (Liu, 2014): An uncertain process X_t is said to have an uncertainty distribution $\Phi_t(x)$ if at each time t, the uncertain variable X_t has the uncertainty distribution $\Phi_t(x)$.

Theorem 1 (Liu, 2014): (Sufficient and Necessary Condition) A function $\Phi_t^{-1}(\alpha): T \times (0,1) \to \Re$ is an inverse uncertainty distribution of independent uncertain process if and only if 1) at each time t, $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function; and 2) for any times $t_2<t_1$, $\Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha)$ is a monotone increasing function with respect to α.

3 METHODOLOGY

In this section, we use an uncertain process called the general geometric Liu process (Liu, 2015) to for ADT modeling, derive the reliability and lifetime distributions, and gives the corresponding statistical analysis method.

3.1 *Accelerated degradation modeling*

In practical applications, the degradation process is usually positive. To describe the positive degradation process under the small sample situation, we consider the following uncertain process

$$X(t) = \exp(e \cdot t + \sigma/\sqrt{t}\, C(t)), \tag{5}$$

where e is the log-drift parameter, also known as the degradation rate. $\sigma/\sqrt{t}$ is the log-diffusion parameter. $C(t)$ is the Liu process that follows a normal uncertainty distribution (N_u) with mean 0 and variance t^2, i.e. $C(t) \sim N_u(0, t)$. Eq.(5) is called the general geometric Liu process. Note that $X(t)$ in Eq.(5) follows a lognormal uncertainty distribution (Log_u), i.e., $X(t) \sim Log_u(et, \sigma\sqrt{t})$. Its uncertainty distribution can be expressed as

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(e \cdot t - \ln x)}{\sqrt{3}\sigma\sqrt{t}}\right)\right)^{-1}. \tag{6}$$

In ADT modeling, acceleration models are usually utilized to describe the relationship between the degradation rate and the accelerated stress level, which can be expressed as

$$\ln e(s_i) = a + b \cdot s_i, \tag{7}$$

where a and b are unknown parameters. s_i is the normalized stress level that can be expressed as

$$s_i = \begin{cases} \dfrac{1/S_0 - 1/S_i}{1/S_0 - 1/S_H} & \text{Arrhenius model,} \\ \dfrac{\ln S_i - \ln S_0}{\ln S_H - \ln S_0} & \text{Power law model,} \\ \dfrac{S_i - S_0}{S_H - S_0} & \text{Exponential model.} \end{cases} \tag{8}$$

where S_0 is the normal stress level, S_i is the i^{th} accelerated stress level, and S_H is the highest accelerated stress level. From Eq.(8), it is easy to know that $s_0 = 0$, and $s_H = 1$.

For simplicity, our proposed uncertain accelerated degradation model in Eq.(5) and Eq.(7) is denoted by M_1.The unknown parameters in model M_1 are summarized as $\mathbf{\Omega} = (a, b, \sigma)$, in which a and b are shown in Eq.(7), σ is shown in Eq.(5).

3.2 *First hitting time and reliability distributions of the proposed model*

After getting the proposed model M_1, we need to derive the reliability and lifetime distributions correspondingly.

Define ω as the failure threshold of the degradation process, then the lifetime T can be defined as the first hitting time (FHT) when the degradation process $X(t)$ reaches ω. Liu (2013) defined the FHT of the uncertain process as follows:

$$t_\omega = \inf\{t_\omega \geq 0 \mid X(t) = \omega\}. \tag{9}$$

According to Theorem 3 in (Liu, 2013), the FHT of an independent increment process with a continuous uncertainty distribution at each time can be expressed as follows,

$$\Upsilon(z) = \mathcal{M}\left\{\sup_{0\le t\le z} X(t) \geq \omega\right\} = 1 - \inf_{0\le t\le z} \Phi_t(\omega). \tag{10}$$

Therefore, before deriving the reliability and lifetime distributions, we firstly need to prove that $X(t)$ in Eq.(5) is an independent increment process with a continuous uncertainty distribution at each time.

Proof:

1) From Eq.(6), it is easy to know that $X(t)$ has a continuous uncertainty distribution at each time t.
2) For each $\gamma \in \Gamma$, the uncertainty distribution of $X(t|\gamma)$ can be expressed as

$$\Phi_t(\gamma) = \left(1 + \exp\left(\frac{\pi\left(e(s)\cdot t - \ln\gamma\right)}{\sqrt{3}\sigma\sqrt{t}}\right)\right)^{-1}. \tag{11}$$

Eq.(11) is obvious a continuous function with respect to time t.

3) From Eq.(6), we can get the inverse uncertainty distribution of $X(t)$ as follows,

$$\Phi_t^{-1}(\alpha) = \exp(e(s)\cdot t + \frac{\sigma\sqrt{3t}}{\pi}\ln\frac{\alpha}{1-\alpha}),\ \alpha \in (0,1). \tag{12}$$

At each time t, the derivative of Eq.(12) with respect to α is

$$\left(\Phi_t^{-1}(\alpha)\right)' = \frac{\sigma\sqrt{3t}}{\pi}\exp(e(s)t + \frac{\sigma\sqrt{3t}}{\pi}\ln\frac{\alpha}{1-\alpha}) \times \frac{1}{\alpha(1-\alpha)}. \tag{13}$$

Since $\alpha \in (0,1)$, we can get that $\alpha(1-\alpha) > 0$. It is easy to prove that $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to α.

4) For any times $0 < t_2 < t_1$, to prove $\Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha)$ is a monotone increasing function with respect to α, we need to prove the following condition:

$$\begin{aligned}\Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha) = \exp\left(e(s)\cdot t_1 + \frac{\sigma\sqrt{3t_1}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) \\ - \exp\left(e(s)\cdot t_2 + \frac{\sigma\sqrt{3t_2}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) > 0.\end{aligned} \tag{14}$$

Since exp(•) is a monotone increasing function, Eq.(14) is equivalent to the following condition:

$$e(s)\cdot(t_1 - t_2) + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\left(\sqrt{t_1} - \sqrt{t_2}\right) > 0. \tag{15}$$

According to the given information, it is easy to prove that the condition in Eq.(15) holds. Thus, $\Phi_{t_1}^{-1}(\alpha) - \Phi_{t_2}^{-1}(\alpha)$ is a monotone increasing function with respect to α. Based on **Theorem 1** in section 2, we can prove that $X(t)$ in Eq.(5) is an independent increment process.

From the above analyses, we can get that the uncertain process $X(t)$ in Eq.(5) is an independent increment process with a continuous uncertainty distribution at each time t. Thus, the uncertainty distribution of the FHT of $X(t)$ can be expressed as follows,

$$\begin{aligned}\Upsilon(z) &= 1 - \inf_{0\le t\le z} \Phi_t(\omega) \\ &= 1 - \inf_{0\le t\le z}\left(1 + \exp\left(\frac{\pi\left(e(s)\cdot t - \ln\omega\right)}{\sqrt{3t}\sigma}\right)\right)^{-1} \\ &= \left(1 + \exp\left(\frac{\pi\left(\ln\omega - e(s)\cdot z\right)}{\sqrt{3z}\sigma}\right)\right)^{-1}.\end{aligned} \tag{16}$$

The corresponding reliability distribution is

$$R_B(t) = \mathcal{M}\{t_\omega > t\} = \left(1 + \exp\left(\frac{\pi\left(e(s)\cdot t - \ln\omega\right)}{\sqrt{3t}\sigma}\right)\right)^{-1} \tag{17}$$

where $R_B(t)$ is known as the "belief reliability" with an uncertain measure (Zeng et al., 2013).

Meanwhile, the belief reliable life, i.e. $BL(\alpha)$, can also be derived and expressed as follows

$$BL(\alpha) = \sup_{0 \le t \le z} R_B(t) \ge \alpha = \Upsilon^{-1}(1-\alpha). \quad (18)$$

3.3 *Statistical analysis method for parameter estimations*

With different loading profiles, there are different kinds of ADT plans, including the constant stress accelerated degradation testing (CSADT), the step stress accelerated degradation testing (SSADT), and the progressive stress accelerated degradation testing (PSADT). Here, we provide the statistical analysis method for parameter estimations in CSADT.

Liu (2015) employed the principle of least squares for parameter estimations of uncertain variables. In this section, we also use this method to estimate the unknown parameters of the proposed model M_1.

Suppose that x_{ijk} is the k^{th} observed degradation value for the j^{th} sample under the i^{th} stress level, and t_{ijk} is the corresponding measurement time, i=1, 2, …, K, j=1, 2, …, n_i, k=1, 2,…, m_i, where K is the number of accelerated stress levels, n_i is the sample size under the i^{th} stress level, and m_i is the number of measurements for the j^{th} sample under the i^{th} stress level.

The unknown parameters of the proposed model M_1 is $\boldsymbol{\Omega}= (a, b, \sigma)$. We proposed a two-step statistical analysis method for the parameter estimations: 1) Collecting belief degrees; 2) Estimating unknown parameters. The procedure of the method is shown in Figure 1 and details are shown as follows:

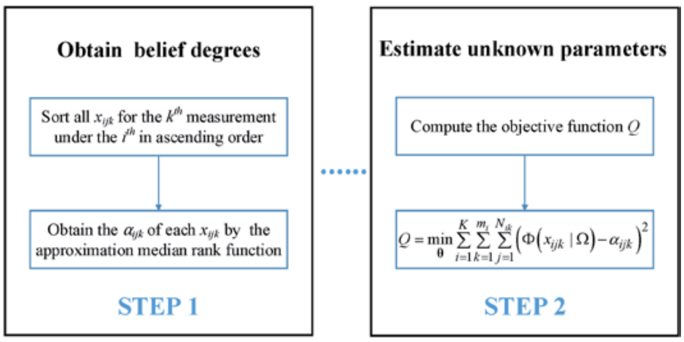

Figure1. The two-step statistical analysis method for parameter estimations of the proposed model.

1) Obtain belief degrees

From Section 3.2, it is known that x_{ik} is an uncertain variable. All the degradation data x_{ijk} of the k^{th} measurement under the i^{th} stress level are the observations of x_{ik}, i.e., x_{ik} = {x_{i1k}, x_{i2k}… x_{ijk}…}, (j=1, 2 … N_i, and n_i is the upper boundary of N_i). Each of the element has a belief degree α_{ijk}. In this section, we use the approximate median rank functions to obtain belief degrees, which is expressed as follows,

$$\alpha_{ijk} = (j-0.3)/(N_{ik}+0.4),\ j = 1,2,\ldots,N_{ik}\,. \quad (19)$$

For all the degradation data of the k^{th} measurement under the i^{th} stress level, if there exist degradation data that are the same, then their belief degrees are also the same.

2) Estimate unknown parameters

According to the principle of least squares, the parameters estimations of the proposed model can be obtained by the principle of least squares, which is

$$Q = \min_{\boldsymbol{\theta}} \sum_{i=1}^{K}\sum_{k=1}^{m_i}\sum_{j=1}^{N_i}\left(\Phi\left(x_{ijk} \mid \Omega\right)-\alpha_{ijk}\right)^2 \quad (20)$$

4 CASE STUDY

In this section, the carbon-film resistors CSADT dataset (Meeker and Escobar, 1998) is used to illustrate the proposed methodology, and discussions are conducted for the sensitivity analysis of the proposed methodology to the sample sizes.

4.1 *The carbon-film resistors CSADT dataset*

In the carbon-film resistors CSADT dataset, there are 9, 10, 10 samples under each accelerated stress levels, which belongs to the small sample situation. Therefore, the proposed methodology can be used to this case for the reliability and lifetime evaluations under normal conditions. Details about this case is shown in Table 1.

Table 1. Basic information about the carbon-film resistors CSADT dataset.

Test information	Contents
Stress levels (temperature/°C)	83, 133, and 173
Normal conditions (°C)	50
Sample size	9, 10, and 10
Measurement times	4, 4, and 4
Failure threshold ω (%)	12

4.2 *Reliability and lifetime evaluations under normal conditions*

Since the accelerated stress is temperature, the Arrhenius model is selected as the acceleration model. Based on the proposed methodology, the parameter estimations are obtained as follows:

Table 2. Parameter estimation results.

Parameters	a	b	σ
Values	-15.18	3.975	9.060e-04

Taking the parameter estimation results in Table 2 into Eq.(17) and Eq.(18), the reliability and lifetime evaluations under normal conditions can be obtained. Results are showed in Figure 2.

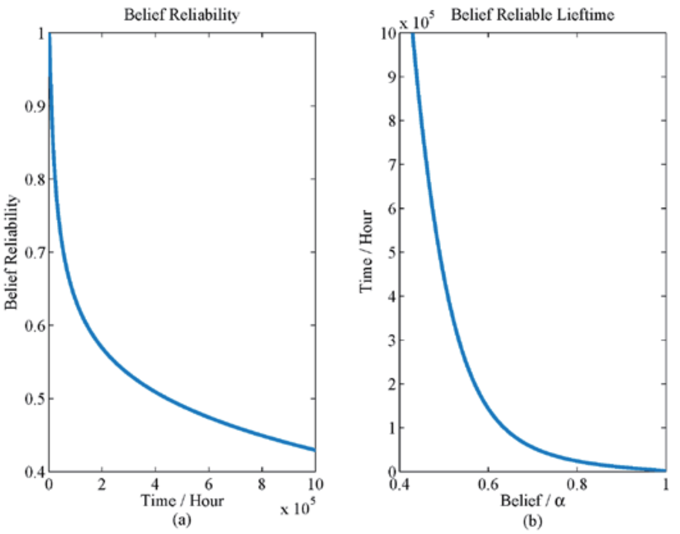

Figure 2. Reliability and lifetime evaluations of the carbon-film resistors ADT dataset under normal conditions.

From Figure 2 (a), it can be seen that the belief reliability changes from the initial value 1 and decreases gradually with the increasing time, which agrees with the intuitive cognition of human beings. If decision makers are interested at belief reliability R_B=0.9, the corresponding belief reliable lifetime BL(0.9)=10181 hours. It means that the belief degree that the products will survived at the normal conditions after 10818 hours is 0.9.

4.3 Discussions

For the sensitivity analysis of the proposed methodology to the sample sizes, we simulate several different situations that has different sample size, and remark each situation as ST_r (n_{r1}, n_{r2}, n_{r3}), r=1, 2,…, 8. ST_r represents the r^{th} situation. n_{r1}, n_{r2}, and n_{r3} represents the chosen sample size under each stress level. Details are shown in Table 3.

Table 3. Different situations for discussions.

Situations | Sample sizes	n_{r1}	n_{r2}	n_{r3}
ST_1	2	3	3
ST_2	3	4	4
ST_3	4	5	5
ST_4	5	6	6
ST_5	6	7	7
ST_6	7	8	8
ST_7	8	9	9
ST_8	9	10	10

As shown in Table 3, under each situation, there are many different combinations of samples, which will lead to many different reliability evaluations. To present the range of reliability evaluations under different situations, under each monitoring time t under the situation ST_r, we choose the minimum and maximum reliability evaluation results as the lower and upper boundaries of the reliability evaluations. So the lower and upper boundaries under each situation can be obtained, and results are shown in Figure 3.

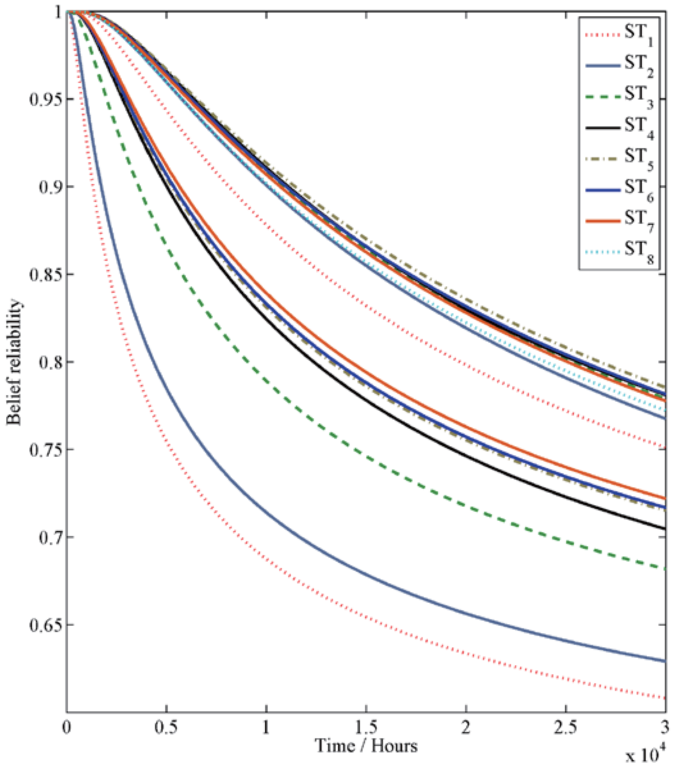

Figure 3. Lower and Upper boundaries of the reliability evaluation results under different sample sizes.

Figure 3 show that with the increasing sample size, the lower and upper boundary are approaching gradually. It indicate that when there are more samples that can provide more information, the epistemic uncertainty in ADT data decreases, which will lead to more stable reliability evaluation results. In addition, the reliability evaluations results under ST_8 is included in the lower and upper boundaries under most situations (ST_3 to ST_7). As for ST_1 and ST_2, the sample size is very small, which makes the provided information too scarce to get stable reliability evaluation results.

The above analysis results show that the proposed methodology is a suitable choice for the small sample situation and can furtherly provide support for the subsequent decision making.

5 CONCLUSIONS

This paper deals with the positive degradation process with small samples in ADT, and concludes as follows,

1) Based on the uncertainty theory, the general geometric Liu process is used to conduct an uncertain accelerated degradation model, which takes the epistemic uncertainty due to small samples in ADT data into consideration.

2) The corresponding statistical analysis method with objective measures is provided for the unknown parameter estimations.
3) The application results show that the reliability evaluation results of the proposed methodology agrees with agrees with the intuitive cognition of human beings, and the discussion results show that the proposed methodology provides stable reliability evaluation results under small samples, which makes it a suitable choice for the small sample situation and can provide support for the subsequent decision making.

In addition to the work of this paper, there are other issues that are worthwhile for future researches. The proposed model is built up on the general geometric Liu process, which is a positive uncertain process. But in practical applications, there are degradation processes which are not only always positive but also strictly monotonic. It is necessary to apply other uncertain processes in ADT to model such degradation processes.

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Some new ranking criteria in data envelopment analysis under uncertain environment

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ABSTRACT

Data Envelopment Analysis (DEA) is a very effective method to evaluate the relative efficiency of decision-making units (DMUs), which has been applied extensively to education, hospital, finance, etc. However, in real-world situations, the data of production processes cannot be precisely measured in some cases, which leads to the research of DEA in uncertain environments. This paper will give some researches to uncertain DEA based on uncertainty theory. Due to the uncertain inputs and outputs, we will give three uncertain DEA models, as well as three types of fully ranking criteria. For each uncertain DEA model, its crisp equivalent model is presented to simplify the computation of uncertain models. Finally, a numerical example is presented to illustrate the three ranking criteria.

1. Introduction

Data envelopment analysis (DEA), as an useful management and decision tool, has been widely used since it was first invented by Charnes, Cooper, and Rhodes (1978). The method is followed by a series of theoretical extensions, such as Banker, Charnes, and Cooper (1984), Charnes, Cooper, Golany, Seiford, and Stutz (1985), Petersen (1990), Tone (2001) and Cooper, Seiford, and Tone (2000). More DEA papers can refer to Seiford (1994) in which 500 references are documented.

In many cases, decision makers are interested in a complete ranking over the dichotomized classification. The researches on ranking have come up for this reason. Over the last decade, many literatures on ranking in DEA have been published. By evaluating DMUs through both self and peer pressure, Sexton, Silkman, and Hogan (1986) can attain a more balanced view of the decision-making units. Andersen and Petersen (1993) developed the super-efficiency approach to get a ranking value which may be greater than one through evaluated DMU's exclusion from the linear constraints. In the benchmark ranking method (Torgersen, Forsund, & Kittelsen, 1996), a DMU is highly ranked if it is chosen as a useful target for many other DMUs.

Most methods of ranking DMUs assume that all inputs and outputs data are exactly known. However, in real situations, such as in a manufacturing system, a production process or a service system, inputs and outputs are volatile and complex so that they are difficult to measure in an accurate way. Thus, people tend to use fuzzy theory to describe the indeterminate inputs and outputs, which motivates the fuzzy DEA. Generally speaking, fuzzy DEA method can be catagorized into four types: the tolerance approach, the α-level based approach, the fuzzy ranking approach and the possibility approach (Adel, Emrouznejad, & Tavana, 2011). In the tolerance approach (refer to Sengupta (1992)), tolerance levels on constraint violations are defined to integrate fuzziness into the DEA models, and the input and output coefficients can be thus treated as crisps. The α-level based approach may be the most popular model of fuzzy DEA. This method discretize the original problem into a series of parametric programs in order to decide the α-cuts of the membership function of efficiency. Related studies include Kao and Liu (2000), Entani, Maeda, and Tanaka (2002), Liu (2008) and Angiz, Emrouznejad, and Mustafa (2012), etc. The fuzzy ranking model is first proposed by Guo and Tanaka (2001), and it focus on determining the fuzzy efficiency scores of DMUs using optimization methods which require ranking fuzzy sets. One can also refer to León, Liern, Ruiz, and Sirvent (2003), Wang and Luo (2006) or Angiz, Tajaddini, Mustafa, and Kamali (2012) for more concepts and information of the fuzzy ranking method. In the possibility approach, the fuzzy DEA models are converted to possibility linear program problem by using possibility

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measures. See Lertworasirikul, Fang, Joines, and Nuttle (2003) for example. Other studies on fuzzy DEA include fuzzy goal programming method (Sheth & Konstantinos, 2003), fuzzy random DEA model in hybrid uncertain environments (Qin & Liu, 2010), fuzzy rough DEA model (Shiraz, Charles, & Jalalzadeh, 2014), cross evaluation approach (Costantino, Dotoli, Epicoco, Falagario, & Sciancalepore, 2012), and fuzzy clustering approach (David & Deep, 2012), etc.

Although the fuzzy DEA models are popular and in most time effective, it may bring some problems to the decision makers in some cases. This is because the possibility measure defined in fuzzy theory doesn't satisfy duality, as explained in Liu (2012). For this reason, an uncertainty theory was founded by Liu (2007) in 2007, and refined by Liu (2010a) in 2010 to deal with the people's belief degree mathematically. A concept of uncertain variable is used to model uncertain quantity, and belief degree is regarded as its uncertainty distribution. As extensions of uncertainty theory, uncertain programming was proposed by Liu (2009) in 2009, which aims to deal with the optimal problems involving uncertain variable. Since then, uncertainty theory was used to solve a variety of real optimal problems, including finance (Chen & Liu, 2010; Peng & Yao, 2010; Liu, 2013), reliability analysis (Liu, 2010b; Zeng, Wen, & Kang, 2013), uncertain graph (Gao, 2013; Gao & Gao, 2013), etc. As an application, this work was followed by uncertain multiobjective programming models, uncertain goal programming models (Liu & Chen, 2013), and uncertain multilevel programming models (Liu & Yao).

In this paper, we will assume the inputs and outputs in DEA models are uncertain variables, and introduce some new DEA models and their ranking criteria based on uncertainty theory. The remainder of this paper is organized as follows: Some basic concept and results on uncertainty theory will be introduced in Section 2; Section 3 will give some basic introduction to DEA models; The method to obtain uncertainty distribution is introduced in Section 4. In Section 5, we will give three uncertain DEA models, three fully ranking criteria, as well as their equivalent deterministic models. Finally, a numerical example will be given to illustrate the uncertain DEA model and the ranking method in Section 6.

2. Preliminaries

Uncertainty theory was founded by Liu (2007) in 2007 and refined by Liu (2010a) in 2010. Nowadays uncertainty theory has become a branch of axiomatic mathematics for modeling human uncertainty. In this section, we will state some basic concepts and results on uncertain variables. These results are crucial for the remainder of this paper.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\} \in [0,1]$. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu (2007, 2010a) presented the three axioms:

(i) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
(ii) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
(iii) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leqslant \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu (2012), thus producing the fourth axiom of uncertainty theory:

(iv) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots, \infty$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}.$$

An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers (Liu, 2007). In order to describe an uncertain variable in practice, the concept of uncertainty distribution is defined as

$$\Phi(x) = \mathcal{M}\{\xi \leqslant x\} \tag{1}$$

for any real number x. For example, the linear uncertain variable $\xi \sim \mathcal{L}(a,b)$ has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leqslant a \\ (x-a)/(b-a), & \text{if } a \leqslant x \leqslant b \\ 1, & \text{if } x \geqslant b. \end{cases} \tag{2}$$

An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leqslant a \\ (x-a)/2(b-a), & \text{if } a \leqslant x \leqslant b \\ (x+c-2b)/2(c-b), & \text{if } b \leqslant x \leqslant c \\ 1, & \text{if } x \geqslant c \end{cases} \tag{3}$$

denoted by $\mathcal{Z}(a,b,c)$ where a, b, c are real numbers with $a < b < c$. An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1} \tag{4}$$

denoted by $\mathcal{N}(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$. An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\} \tag{5}$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Theorem 1 (Liu, 2010a). *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{6}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1} = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)). \tag{7}$$

Theorem 2 (Liu & Ha, 2010). *Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \tag{8}$$

provided that $E[\xi]$ exists.

3. DEA model

CCR model is one of the most frequently used DEA model, which was proposed by Charnes et al. (1978). Since the following sections will use this model, we will give some basic introduction to CCR model. Firstly let us review some symbols and variables:

DMU_k: the kth DMU, $k = 1, 2, \ldots, n$;
DMU_0: the target DMU;
$\boldsymbol{x}_k \in R^{p\times 1}$: the inputs vector of DMU_k, $k = 1, 2, \ldots, n$;
$\boldsymbol{x}_0 \in R^{p\times 1}$: the inputs vector of the target DMU_0;
$\boldsymbol{y}_k \in R^{q\times 1}$: the outputs vector of DMU_i, $k = 1, 2, \ldots, n$;
$\boldsymbol{y}_0 \in R^{q\times 1}$: the outputs vector of the target DMU_0;
$\boldsymbol{u} \in R^{p\times 1}$: the vector of input weights;
$\boldsymbol{v} \in R^{q\times 1}$: the vector of output weights.

In this model, the efficiency of entity evaluated is obtained as a ratio of the weighted output to the weighted input subject to the condition that the ratio for every entity is not larger than 1. Mathematically, it is described as follows:

$$\begin{aligned} &\max_{\boldsymbol{u},\boldsymbol{v}}\ \theta = \frac{\boldsymbol{v}^T\boldsymbol{y}_0}{\boldsymbol{u}^T\boldsymbol{x}_0} \\ &\text{subject to :} \\ &\qquad \boldsymbol{v}^T\boldsymbol{y}_j \leqslant \boldsymbol{u}^T\boldsymbol{x}_j,\ j = 1, 2, \ldots, n \\ &\qquad \boldsymbol{u} \geqslant 0 \\ &\qquad \boldsymbol{v} \geqslant 0. \end{aligned} \tag{9}$$

Definition 1 (*Efficiency*). DMU_0 is efficient if $\theta^* = 1$, where θ^* is the optimal value of (9).

4. Acquisition method of uncertainty distribution

In uncertain DEA models, which will be introduced in Section 5, inputs and outputs are regarded as uncertain variables. A key problem is to determine their uncertainty distributions. Different from the method to determine probability distributions in probability theory, uncertainty distributions cannot be obtained by historical data since the sample size is too small (even no samples). Thus, we have to invite some domain experts to evaluate their belief degrees of each event will occur. For this purpose, Liu (2010a) proposed a questionnaire survey method for collecting expert's experimental data and determine the uncertainty distribution.

The starting point is to invite one expert who is asked to complete a questionnaire about the value of some uncertain input or output variable ξ like "What is the value of input variable ξ?".

We first ask the domain expert to choose a possible value x that the uncertain demand ξ may take, and then quiz him"How likely is ξ less than or equal to x?".

Denote the expert's belief degree by α. An expert's experimental data (x, α) thus acquired from the domain expert.

Repeating the above process, we can obtain the following expert's experimental data

$$(x_1, \alpha_1),\ (x_2, \alpha_2), \ldots, (x_n, \alpha_n) \tag{10}$$

that meet the following consistence condition (perhaps after a rearrangement)

$$x_1 < x_2 < \ldots < x_n,\ 0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \alpha_n \leqslant 1. \tag{11}$$

Based on those expert's experimental data, Liu (2010a) suggested an empirical uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leqslant x_1 \\ \alpha_i + \frac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{x_{i+1}-x_i}, & \text{if } x_i \leqslant x \leqslant x_{i+1},\ 1 \leqslant i < n \\ 1, & \text{if } x > x_n. \end{cases} \tag{12}$$

denoted by $\mathcal{E}(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_n, \alpha_n)$. Essentially, it is a type of linear interpolation method.

Assume there are m domain experts and each produces an uncertainty distribution. Then we may get m uncertainty distributions $\Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x)$. The Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. Wang, Gao, and Guo (2012) recast the Delphi method as a process to determine the uncertainty distribution. The main steps are listed as follows:

Step 1: The m domain experts provide their expert's experimental data,

$$(x_{ij}, \alpha_{ij}), \quad j = 1, 2, \ldots, n_i,\ i = 1, 2, \ldots, m. \tag{13}$$

Step 2: Use the i-th expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \ldots, (x_{in_i}, \alpha_{in_i})$ to generate the i-th expert's uncertainty distribution Φ_i.
Step 3: Compute $\Phi(x) = w_1\Phi_1(x) + w_2\Phi_2(x) + \ldots + w_m\Phi_m(x)$ where $w_1, w_2, \ldots, w_m$ are convex combination coefficients.
Step 4: If $|\alpha_{ij} - \Phi(x_{ij})|$ are less than a given level $\varepsilon > 0$, then go to Step 5. Otherwise, the i-th expert receives the summary (Φ and reasons), and then provides a set of revised expert's experimental data. Go to Step 2.
Step 5: The last Φ is the uncertainty distribution of the customer's demand.

5. Uncertain DEA ranking criteria

This section will give some researches to empirical uncertain DEA based on uncertainty theory introduced in Section 2. The new symbols and notations are given as follows:

$\tilde{\boldsymbol{x}}_k = (\tilde{x}_{k1}, \tilde{x}_{k2}, \ldots, \tilde{x}_{kp})$: the uncertain input vectors of DMU_k, $k = 1, 2, \ldots, n$;
$\tilde{\boldsymbol{y}}_k = (\tilde{y}_{k1}, \tilde{y}_{k2}, \ldots, \tilde{y}_{kq})$: the uncertain output vectors of DMU_k, $k = 1, 2, \ldots, n$;
$\boldsymbol{\Phi}_k(x) = (\Phi_{k1}(x), \Phi_{k2}(x), \ldots, \Phi_{kp}(x))$: the uncertainty distribution vector of $\tilde{\boldsymbol{x}}_k = (\tilde{x}_{k1}, \tilde{x}_{k2}, \ldots, \tilde{x}_{kp}), k = 1, 2, \ldots, n$;
$\boldsymbol{\Psi}_k(x) = (\Psi_{k1}(x), \Psi_{k2}(x), \ldots, \Psi_{kq}(x))$: the uncertainty distribution vector of $\tilde{\boldsymbol{y}}_k = (\tilde{y}_{k1}, \tilde{y}_{k2}, \ldots, \tilde{y}_{kq}), k = 1, 2, \ldots, n$.

In the following sections, three types of uncertain DEA fully ranking criteria are to be investigated.

5.1. The expected ranking criterion

Liu (2007, 2012) proposed the expected value operator of uncertain variable and uncertain expected value model. The essential idea of the uncertain expected DEA model is to optimize the expected value of $\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0}$ subject to some chance constraints, then we have the first type of the uncertain DEA model:

$$\begin{cases} \theta = \max\limits_{\boldsymbol{u},\boldsymbol{v}} \mathrm{E}\left[\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0}\right] \\ \text{subject to :} \\ \qquad \mathcal{M}\{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_k \leqslant \boldsymbol{u}^T\tilde{\boldsymbol{x}}_k\} \geqslant \alpha,\ k = 1, 2, \ldots, n \\ \qquad \boldsymbol{u} \geqslant 0 \\ \qquad \boldsymbol{v} \geqslant 0 \end{cases} \tag{14}$$

in which $\alpha \in (0.5, 1]$.

Definition 2. A vector $(\boldsymbol{u}, \boldsymbol{v}) \geqslant 0$ is called a feasible solution to the uncertain programming model (14) if

$$\mathcal{M}\{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_k \leqslant \boldsymbol{u}^T\tilde{\boldsymbol{x}}_k\} \geqslant \alpha \tag{15}$$

for $k = 1, 2, \ldots, n$.

Definition 3. A feasible solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is called an expected optimal solution to the uncertain programming model (14) if

$$E\left[\frac{\boldsymbol{v}^{*T}\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^{*T}\tilde{\boldsymbol{x}}_0}\right] \geqslant E\left[\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0}\right] \tag{16}$$

for any feasible solution $(\boldsymbol{u}, \boldsymbol{v})$.

Expected Ranking Criterion: The greater the optimal objective value is, the more efficient DMU_0 is ranked.

Theorem 3. *Assume that $\tilde{x}_{1i}, \tilde{x}_{2i}, \ldots, \tilde{x}_{ni}$ are independent uncertain inputs with uncertainty distribution $\Phi_{1i}, \Phi_{2i}, \ldots, \Phi_{ni}$ for each i, $i = 1, 2, \ldots, p$, and $\tilde{y}_{1i}, \tilde{y}_{2i}, \ldots, \tilde{y}_{ni}$ are independent uncertain outputs with uncertainty distribution $\Psi_{1j}, \Psi_{2j}, \ldots, \Psi_{nj}$ for each j, $j = 1, 2, \ldots, q$. Then the uncertain programming model (14) is equivalent to the following model:*

$$\begin{cases} \theta = \max\limits_{\boldsymbol{u},\boldsymbol{v}} \int_0^1 \frac{\boldsymbol{v}^T\Psi_0^{-1}(\alpha)}{\boldsymbol{u}^T\Phi_0^{-1}(1-\alpha)} d\alpha. \\ \text{subject to :} \\ \qquad \boldsymbol{v}^T\Psi_k^{-1}(\alpha) \leqslant \boldsymbol{u}^T\Phi_k^{-1}(1-\alpha), \; k = 1, 2, \ldots, n \\ \qquad \boldsymbol{u} \geqslant 0 \\ \qquad \boldsymbol{v} \geqslant 0. \end{cases} \tag{17}$$

Proof. Since the function $\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}}$ is strictly increasing with respect to $\tilde{\boldsymbol{y}}$ and strictly decreasing with respect to $\tilde{\boldsymbol{x}}$, it follows from Theorem 1 that the inverse uncertainty distribution of $\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}}$ is $\frac{\boldsymbol{v}^T\Psi^{-1}(\alpha)}{\boldsymbol{u}^T\Phi^{-1}(1-\alpha)}$. Thus $\mathcal{M}\{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_k \leqslant \boldsymbol{u}^T\tilde{\boldsymbol{x}}_k\} \geqslant \alpha$ holds if and only if $\boldsymbol{v}^T\Psi_k^{-1}(\alpha) \leqslant \boldsymbol{u}^T\Phi_k^{-1}(1-\alpha)$ for $k = 1, 2, \ldots, n$. By using Theorem 2, we obtain

$$\text{E}\left[\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0}\right] = \int_0^1 \frac{\boldsymbol{v}^T\Psi_0^{-1}(\alpha)}{\boldsymbol{u}^T\Phi_0^{-1}(1-\alpha)} d\alpha. \tag{18}$$

The theorem is thus verified. □

5.2. The optimistic ranking criterion

Chance-constrained programming (CCP), which was initialized by Charnes and Cooper (1961), offers a powerful means for modelling stochastic decision systems. The essential idea of chance-constrained programming is to optimize some critical value with a given confidence level subject to some chance constraints. Inspired by this idea, Liu (2010a) extended it to uncertain programming models. Assuming that the decision makers want to maximize the optimistic value of the uncertain objective at given confidence level, we have the second type of DEA model:

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \bar{f} \\ \text{subject to :} \\ \qquad \mathcal{M}\left\{\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \geqslant \bar{f}\right\} \geqslant 1-\alpha \\ \qquad \mathcal{M}\{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_k \leqslant \boldsymbol{u}^T\tilde{\boldsymbol{x}}_k\} \geqslant \alpha, \; k = 1, 2, \ldots, n \\ \qquad \boldsymbol{u} \geqslant 0 \\ \qquad \boldsymbol{v} \geqslant 0 \end{cases} \tag{19}$$

in which $\alpha \in (0.5, 1]$.

Definition 4. A feasible solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is called an optimistic optimal solution to the uncertain programming model (19) if

$$\max\left\{\bar{f} \mid \mathcal{M}\left\{\frac{\boldsymbol{v}^{*T}\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^{*T}\tilde{\boldsymbol{x}}_0} \geqslant \bar{f}\right\} \geqslant 1-\alpha\right\} \geqslant \max\left\{\bar{f} \mid \mathcal{M}\left\{\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \geqslant \bar{f}\right\} \geqslant 1-\alpha\right\} \tag{20}$$

for any feasible solution $(\boldsymbol{u}, \boldsymbol{v})$.

Optimistic Ranking Criterion: The greater the optimal objective value is, the more efficient DMU_0 is ranked.

Theorem 4. *Assume that $\tilde{x}_{1i}, \tilde{x}_{2i}, \ldots, \tilde{x}_{ni}$ are independent uncertain inputs with uncertainty distribution $\Phi_{1i}, \Phi_{2i}, \ldots, \Phi_{ni}$ for each i, $i = 1, 2, \ldots, p$, and $\tilde{y}_{1i}, \tilde{y}_{2i}, \ldots, \tilde{y}_{ni}$ are independent uncertain outputs with uncertainty distribution $\Psi_{1j}, \Psi_{2j}, \ldots, \Psi_{nj}$ for each j, $j = 1, 2, \ldots, q$. Then the uncertain programming model (19) is equivalent to the following model:*

$$\begin{cases} \max\limits_{\boldsymbol{u},\boldsymbol{v}} \frac{\boldsymbol{v}^T\Psi_0^{-1}(\alpha)}{\boldsymbol{u}^T\Phi_0^{-1}(1-\alpha)} \\ \text{subject to :} \\ \qquad \boldsymbol{v}^T\Psi_k^{-1}(\alpha) \leqslant \boldsymbol{u}^T\Phi_k^{-1}(1-\alpha), \; k = 1, 2, \ldots, n \\ \qquad \boldsymbol{u} \geqslant 0 \\ \qquad \boldsymbol{v} \geqslant 0. \end{cases} \tag{21}$$

Proof. By using Theorem 1, the theorem can be easily obtained. □

5.3. The maximal chance ranking criterion

Sometimes the decision maker may want to maximize the chance of satisfying the event $\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \geqslant 1$. In order to model this type of decision system, Liu (1997), Liu (1999) and Liu (2002) provided the dependent-chance programming (DCP). Here we carried out the DCP model into the DEA as follows:

$$\begin{cases} \theta = \max\limits_{\boldsymbol{u},\boldsymbol{v}} \mathcal{M}\left\{\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \geqslant 1\right\} \\ \text{subject to :} \\ \qquad \mathcal{M}\{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_k \leqslant \boldsymbol{u}^T\tilde{\boldsymbol{x}}_k\} \geqslant \alpha, \; k = 1, 2, \ldots, n \\ \qquad \boldsymbol{u} \geqslant 0 \\ \qquad \boldsymbol{v} \geqslant 0 \end{cases} \tag{22}$$

in which $\alpha \in (0.5, 1]$.

Definition 5. A feasible solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is called an maximal chance optimal solution to the uncertain programming model (22) if

$$\mathcal{M}\left\{\frac{\boldsymbol{v}^{*T}\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^{*T}\tilde{\boldsymbol{x}}_0} \geqslant 1\right\} \geqslant \mathcal{M}\left\{\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \geqslant 1\right\} \tag{23}$$

for any feasible solution $(\boldsymbol{u}, \boldsymbol{v})$.

Maximal Chance Ranking Criterion: The greater the optimal objective value is, the more efficient DMU_0 is ranked.

Theorem 5. *Assume that $\tilde{x}_{1i}, \tilde{x}_{2i}, \ldots, \tilde{x}_{ni}$ are independent uncertain inputs with uncertainty distribution $\Phi_{1i}, \Phi_{2i}, \ldots, \Phi_{ni}$ for each i, $i = 1, 2, \ldots, p$, and $\tilde{y}_{1i}, \tilde{y}_{2i}, \ldots, \tilde{y}_{ni}$ are independent uncertain outputs with uncertainty distribution $\Psi_{1j}, \Psi_{2j}, \ldots, \Psi_{nj}$ for each j, $j = 1, 2, \ldots, q$. Then the uncertain programming model (22) is equivalent to the following model:*

Table 1
DMUs with two uncertain inputs and two uncertain outputs.

DMU_i	1	2	3	4	5
Input 1	$\mathcal{Z}(3.5,4.0,4.5)$	$\mathcal{Z}(2.9,2.9,2.9)$	$\mathcal{Z}(4.4,4.9,5.4)$	$\mathcal{Z}(3.4,4.1,4.8)$	$\mathcal{Z}(5.9,6.5,7.1)$
Input 2	$\mathcal{Z}(2.9,3.1,3.3)$	$\mathcal{Z}(1.4,1.5,1.6)$	$\mathcal{Z}(3.2,3.6,4.0)$	$\mathcal{Z}(2.1,2.3,2.5)$	$\mathcal{Z}(3.6,4.1,4.6)$
Output 1	$\mathcal{Z}(2.4,2.6,2.8)$	$\mathcal{Z}(2.2,2.2,2.2)$	$\mathcal{Z}(2.7,3.2,3.7)$	$\mathcal{Z}(2.5,2.9,3.3)$	$\mathcal{Z}(4.4,5.1,5.8)$
Output 2	$\mathcal{Z}(3.8,4.1,4.4)$	$\mathcal{Z}(3.3,3.5,3.7)$	$\mathcal{Z}(4.3,5.1,5.9)$	$\mathcal{Z}(5.5,5.7,5.9)$	$\mathcal{Z}(6.5,7.4,8.3)$

Table 2
Expected ranking results with different α.

Confidence level α	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0.5	0.82	1	0.91	1	1
0.6	0.85	1	0.92	1	1
0.7	0.89	1	0.94	1	1
0.8	0.90	1	0.98	1	1
0.9	0.91	1	1	1	1

Table 3
Optimistic ranking results with different α.

Confidence level α	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0.5	0.82	1	0.89	1	1
0.6	0.85	1	0.91	1	1
0.7	0.89	1	0.94	1	1
0.8	0.90	1	0.98	1	1
0.9	0.91	0.99	1	1	1

Table 4
Maximal ranking results with different α.

Confidence level α	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0.5	0.10	0.50	0.31	0.50	0.50
0.6	0.06	0.40	0.26	0.40	0.40
0.7	0.03	0.30	0.22	0.30	0.30
0.8	0	0.20	0.19	0.20	0.20
0.9	0	0.08	0.10	0.10	0.10

Table 5
Ranking results for different criteria.

Ranking criteria	$\alpha = 0.5,\ 0.6,\ 0.7,\ 0.8$	$\alpha = 0.9$
Expected ranking criterion	DMU_2, DMU_4, DMU_5, DMU_3, DMU_1	DMU_3, DMU_4, DMU_5, DMU_2, DMU_1
Optimistic ranking criterion	DMU_2, DMU_4, DMU_5, DMU_3, DMU_1	DMU_3, DMU_4, DMU_5, DMU_2, DMU_1
Maximal chance ranking criterion	DMU_2, DMU_4, DMU_5, DMU_3, DMU_1	DMU_3, DMU_4, DMU_5, DMU_2, DMU_1

$$\begin{cases} \theta = \max\limits_{\boldsymbol{u},\boldsymbol{v}} \mathcal{M}\left\{\frac{\boldsymbol{v}^T\tilde{\boldsymbol{y}}_0}{\boldsymbol{u}^T\tilde{\boldsymbol{x}}_0} \geqslant 1\right\} \\ \text{subject to :} \\ \qquad \boldsymbol{v}^T\boldsymbol{\Psi}_k^{-1}(\alpha) \leqslant \boldsymbol{u}^T\boldsymbol{\Phi}_k^{-1}(1-\alpha),\ k = 1,2,\ldots,n \\ \qquad \boldsymbol{u} \geqslant 0 \\ \qquad \boldsymbol{v} \geqslant 0. \end{cases} \tag{24}$$

Proof. By using Theorem 1, the theorem can be easily obtained. □

6. A numerical example

This example aims to illustrate the three uncertain DEA models and their corresponding ranking methods. For simplicity, we will only consider five DMUs with two inputs and two outputs which are all zigzag uncertain variables denoted by $\mathcal{Z}(a,b,c)$. Table 1 gives the information of the DMUs.

From Tables 2–4, we can get the following conclusions:

(i) Roughly speaking, the ranking results are DMU_2, DMU_4, DMU_5, DMU_3, DMU_1.
(ii) As shown in Table 5, the confidence level α affects the ranking results. When $\alpha = 0.90$, the DMUs are ranked: DMU_3, DMU_4, DMU_5, DMU_2, DMU_1. At other α, the DMUs are ranked: DMU_2, DMU_4, DMU_5, DMU_3, DMU_1; This phenomena indicates that the ranking method in uncertain environment is more complex than the traditional ranking methods because of the inherent uncertainty contained in inputs and outputs.
(iii) Although the ranking results with different ranking criterion are uniform in this example, the three ranking criterion are different in nature.

The results of developed ranking criteria are then compared with those that are obtained from a fuzzy DEA model introduced by Guo and Tanaka (2001). The fuzzy DEA model only gives whether the DMUs are efficient under different possibility levels h, as shown in Table 6. Here we only select the results of the max-

Table 6
Evaluating results with different *h* from Guo and Tanaka (2001).

h	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0	Inefficient	Efficient	Efficient	Efficient	Efficient
0.5	Inefficient	Efficient	Inefficient	Inefficient	Efficient
0.75	Inefficient	Efficient	Inefficient	Inefficient	Efficient
1	Inefficient	Efficient	Inefficient	Efficient	Efficient

Table 7
Efficiency of DMUs in maximal ranking results.

α	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0.5	Inefficient	Efficient	Inefficient	Efficient	Efficient
0.6	Inefficient	Efficient	Inefficient	Efficient	Efficient
0.7	Inefficient	Efficient	Inefficient	Efficient	Efficient
0.8	Inefficient	Efficient	Inefficient	Efficient	Efficient
0.9	Inefficient	Inefficient	Inefficient	Efficient	Efficient

imal ranking criteria, and give the efficiency of these DMUs for comparison (see Table 7). We can find from the two tables that most efficiency properties of DMUs in our developed method are consistent with those derived from the fuzzy DEA model. However, the differencies between two results also indicate our preferences of evaluating DMUs are not exactly the same under different theoretical basis.

7. Conclusion

Due to its widely practical used background, data envelopment analysis (DEA) has become a pop area of research. Since the data cannot be precisely measured in some practical cases, many paper have been published when the inputs and outputs are uncertain. This paper gave some researches to uncertain DEA based on uncertainty measure. Three uncertain DEA models have been proposed, which leaded to three fully ranking criteria. In order to simplify the computation of the uncertain DEA model, we have presented their equivalent crisp models. The numerical example illustrated the uncertain DEA models and the ranking methods.

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An optimal evaluating method for uncertainty metrics in reliability based on uncertain data envelopment analysis

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ABSTRACT

In reliability engineering, there always exist uncertainties caused by the scarcity of data and information. Various metrics are developed to measure reliability under uncertainty, including evidence-theory-based reliability, interval-analysis-based reliability, fuzzy-interval-analysis-based reliability, posbist reliability and belief reliability. As these five metrics are difficult to be fully understood, it is hard to select the most appropriate one for a specific condition. This paper will propose an uncertain evaluating model to conduct an objective evaluation on these uncertainty metrics. An evaluating index system is established from the aspects of capabilities and adaptabilities. Uncertainty theory is adopted to deal with subjective uncertainties in the quantification process of the index system. Then an evaluating method based on uncertain data envelopment analysis is proposed to provide decision-makers with a succinct result for a given operational context. Finally, the evaluating method is illustrated with a numerical example, which shows that final metric choices vary with different requirements.

1. Introduction

Due to the lack of data and information, uncertainties have a great influence on the results of evaluation in reliability engineering. Various metrics have been developed to measure reliability under uncertainty. In this paper we consider evidence-theory-based reliability (EB reliability) [1–3], interval-analysis-based reliability (IB reliability) [4–6], fuzzy-interval-analysis-based reliability (FIB reliability) [7,8], posbist reliability [9,10] and belief reliability [11,12].

EB reliability, IB reliability, FIB reliability are characterized as probability-interval-based metrics. They differ from each other with the method to construct the reliability intervals [13]. EB reliability is widely used in reliability measurement for its flexible axiom system and its ability to allow epistemic uncertainty and aleatory uncertainty to be treated separately within a single framework without any assumptions [14]. IB reliability can be used as a Probability Bounds (P-Box) to calculate the maximum and minimum of failure probability with the range of input parameters and is especially effective in situations in which one cannot specify parameter values for input distributions, precise probability distributions (shape) and dependencies between input parameters [15]. IB reliability has been used to model complex systems with limited information for static and dynamic reliability problems [16,17]. FIB reliability, based on the Extension Principle of Zadeh [18], allows the consideration of both aleatory and epistemic uncertainty simultaneously. In fuzzy problems, it is important to approximate the expected values, and Li developed an effective algorithm to approximate expected values for ordinary fuzzy problems [19]. In practice, FIB reliability can be classified in two ways. Beer [8] regarded FIB reliability as a combination of probability theory and fuzzy set theory, where aleatory uncertainty and subjective probabilistic information were captured in probabilistic models while imprecision in the probabilistic model specification was described with fuzzy sets. On the other hand, Aven et al. [7] described aleatory uncertainty by using probability distributions and epistemic uncertainty by using the possibility distributions in the framework of fuzzy set theory. Posbist reliability, established by Cai, is based on the possibility theory [20]. Similar to EB reliability, posbist reliability also allows epistemic uncertainty and aleatory uncertainty to be treated separately within a single framework [14]. Posbist reliability is used to solve reliability of typical system structures [21,22] and reliability of k-out-of-n system structures [23]. Uncertainty theory was established by Liu as a branch of axiomatic mathematics for modeling human uncertainty based on normality, duality, subadditivity and product axioms [24]. Belief reliability is based on uncertainty theory and is used as the uncertainty measure of the system to perform specified functions within given time under given operating conditions. The influences of design margin, aleatory uncertainty and epistemic uncertainty are considered in the evaluation framework for component belief reliability [25].

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The five metrics have different desirable and undesirable features from the aspects of people's demands and preferences [13]. Since there is no historical data available to engineers, these features, demands and preferences can be measured by knowledge and experience of domain experts, which are subjective and uncertain. In 2007, Liu [24] founded an uncertainty theory to deal with human's subjective uncertainty by belief degree mathematically and in 2010, Liu [26] refined it based on normality, duality, subadditivity and product measure axioms. Therefore, we can regard features as uncertain variables and treat demands and preferences as belief degrees.

Following the quantification of subjective uncertainties, a key point is how to build a model to find the optimal solution in an objective way. To deal with the optimal problems involving uncertain variables, Liu proposed uncertain programming in 2009 [27]. Following uncertain multi-objective programming [28], uncertain goal programming [28] and uncertain multilevel programming [29], Wen et al. proposed a data envelopment analysis (DEA) model in an uncertain environment in 2014 [30]. DEA is a data oriented approach for evaluating the performance of a set of peer entities called Decision Making Units (DMUs) [31]. The definition of a DMU is generic and flexible. Because it requires very few assumptions, DEA can also be implemented in cases with complex nature. Thus, we will propose an optimal evaluating method based on uncertainty theory and DEA model.

The remainder of this paper is organized as follows. Section 2 introduces some basic concepts about uncertainty theory. Section 3 describes the evaluation indexes and the quantification process. In Section 4, an uncertain evaluating model based on uncertain DEA method is proposed to assist the selection of uncertainty metrics. This model is subsequently applied in Section 5 to select from the five uncertainty reliability metrics based on different requirements.

2. Preliminaries

Uncertainty theory was founded by Liu in 2007 [24] and refined by Liu in 2010 [26]. Following that, uncertain process [32], uncertain differential equations [32], uncertain calculus [33] and uncertain programming [27] were proposed. Uncertainty theory has been successfully applied to solve various problems, including finance [34], reliability [35] and graph [36]. In this section, we will state some basic concepts and results on uncertain variables. These results are crucial for the remainder of this paper.

Let Γ be a nonempty set, and let be a σ-algebra over Γ. Each element Λ *in* is called an event. Liu [24] defined an uncertain measure by the following axioms:

Axiom 1. (Normality axiom)$\{\Gamma\}=1$ for the universal set Γ.

Axiom 2. (Duality axiom) $\{\Lambda\}+\{\Lambda^C\}=1$for any event Λ.

Axiom 3. (Subadditivity axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}\{\Lambda_i\}. \tag{1}$$

Furthermore, Liu [33] defined a product uncertain measure by the fourth axiom:

Axiom 4. (Product axiom) Let $(\Gamma_i, {}_i, {}_i)$ be uncertain spaces for $i=1,2,\cdots$. The product uncertain measure is an uncertain measure satisfying

$$\left\{\prod_{i=1}^{\infty}\Lambda_i\right\}=\bigwedge_{i=1}^{\infty}{}_i\{\Lambda_i\} \tag{2}$$

where $_i$ are σ-algebras over Γ_i, and $\Lambda_i$are arbitrarily chosen events from $_i$ for $i=1,2,\cdots$, respectively.

Definition 1. (see Liu [24]). Let Γ be a nonempty set, let be a σ-algebra over Γ, and let be an uncertain measure. Then the triplet $(\Gamma, ,)$ is called an uncertainty space.

Definition 2. (see Liu [24]). An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, ,)$ to the set of real numbers, i.e., for any Borel set B of real numbers, we have

$$\{\xi\in B\}=\{\gamma\in\Gamma|\xi(\gamma)\in B\}\in. \tag{3}$$

Definition 3. (see Liu [24]). The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x)=\{\xi\le x\} \tag{4}$$

for any real number x.

Example 4. An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x)=\left(1+\exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1},\quad x\in \tag{5}$$

denoted by $\mathcal{N}(e,\sigma)$ where e and σ are real numbers with $\sigma>0$.

Definition 5. (see Liu [24]). Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Example 6. The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(e,\sigma)$ is

$$\Phi_x^{-1}(\alpha)=e+\frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{6}$$

3. Evaluation indexes

3.1. Determination of evaluation indexes

In reliability engineering, engineers hope that the method is complete in theory and practical in application. Theoretical completeness determines types of problem a metric can solve and forms of conclusion a metric can draw, while engineering practicability reflects difficulty and complexity when utilizing a metric.

In this paper, six types of adaptabilities are used to describe theoretical completeness and three types of capabilities are used to represent engineering practicability. The index system consisting of adaptabilities and capabilities is tabulated in Table 1.

3.2. Quantification of evaluation indexes

Both adaptabilities and capabilities can merely be measured by the knowledge and experience of domain experts. To quantify expert's knowledge and experience, this paper adopts a questionnaire survey process [37] to convert knowledge and experience into concrete data. An example is delivered to interpret this process.

Table 1
Description of index system.

Notation	Meaning	Description
A_1	Adaptability to mixed uncertainty	Measures how well the metric can adapt to the uncertainty problem which solves aleatory and epistemic uncertainty in an unseparated way.
A_2	Adaptability to separate uncertainty	Measures how well the metric can adapt to the uncertainty problem which solves aleatory and epistemic uncertainty separately.
A_3	Adaptability to big data	Measures how well the metric can adapt to the uncertainty problem in terms of big data.
A_4	Adaptability to spare data	Measures how well the metric can adapt to the uncertainty problem in terms of spare data.
A_5	Adaptability to duality axiom	Measures whether the metric can satisfy the duality axiom.
A_6	Adaptability to slow attenuation	Measures whether the metric can compensate the conservatism in the estimations of the component-level reliability metrics.
C_1	Learning difficulty	Measures how difficult it is for engineers to learn the candidate metrics.
C_2	Application difficulty	Measures how difficult it is for engineers to apply the candidate metrics.
C_3	Complexity	Measures how complex it is in computation process.

Example 7. The consultation process of evaluating adaptability to spare data.

Q1: The score of EB reliability's adaptability to spare data is from 0 to 100, what do you think is a likely score?

A1: 90.

Q2: To what extend do you think that evidence theory's adaptability to spare data is less than 90?

A2: 80%. (an expert's experimental data (90,0.8) is acquired).

Q3: Is there another number this score may be?

A3: 70.

Q4: To what extend do you think that evidence theory's adaptability to spare data is less than 70?

A4: 50%. (an expert's experimental data (70,0.5) is acquired).

Q5: Is there another number this score may be? If yes, what is it?

A5: 40.

Q6: To what extend do you think that evidence theory's adaptability to spare data is less than 40?

A6: 40%. (an expert's experimental data (40,0.4) is acquired).

Q7: Is there another number this score may be? If yes, what is it?

A7: 10.

Q8: To what extend do you think that evidence theory's adaptability to spare data is less than 10?

A8: 5%. (an expert's experimental data (10,0.05) is acquired).

Q9: Is there another number this score may be? If yes, what is it?

A9: No idea.

By using the questionnaire survey, four expert's experimental data of evidence theory's adaptability to spare data are acquired from the domain expert,

$$(90, 0.9), (70, 0.5), (40, 0.4), (10, 0.05). \tag{7}$$

Theorem 8. Maximum Entropy Principle (see Chen and Dai [38]) Let ξ be an uncertain variable with finite expected value e and variance σ^2. Then

$$H[\xi] \leq \frac{\pi\sigma}{\sqrt{3}} \tag{8}$$

and the equality holds if ξ is a normal uncertain variable with expected value e and variance σ^2, i.e., $\mathcal{N}(e, \sigma)$.

According to the maximum entropy principle, we assume that the adaptabilities and capabilities follow normal uncertainty distributions.

Principle of least squares [26] can be adopted to determine the uncertainty distributions of uncertain variable $\Phi(x)$.

Definition 9. Principle of least squares (see Liu [26]) If the expert's experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n) \tag{9}$$

are obtained and the vertical direction is accepted, then we have

$$\min_{\theta} \sum_{i=1}^{n} (\Phi(x_i|\theta) - \alpha_i)^2. \tag{10}$$

The optimal solution $\hat{\theta}$ of Eq. (10) is called the least squares estimate of θ, and then the least squares uncertainty distribution is $\Phi(x_i|\hat{\theta})$.

4. Uncertainty evaluating model

The uncertain evaluating model in this section is based on the uncertain DEA model [30]. Similar to traditional DEA model [39], the objective of the uncertain DEA model is to maximize the total slacks in inputs and outputs subject to the constraints.

The developed model chooses inputs and outputs from the index system as follows: the inputs of the model are capabilities denoted by C_{ki}, where C_{ki} is *ith* capability of *kth* candidate method evaluated by experts. The outputs of the model are adaptabilities represented by A_{kj}, where A_{kj} is *jth* capability of *kth* candidate method evaluated by experts.

Relative symbols and notations are introduced briefly as follows:

DMU_k: the kth DMU, $k = 1, 2, \cdots, 5$;

DMU_0: the target DMU;

C_{k1}: uncertain variable representing learning difficulty, $k = 1, 2, \cdots, 5$;

C_{k2}: uncertain variable representing application difficulty, $k = 1, 2, \cdots, 5$;

C_{k3}: uncertain variable representing complexity, $k = 1, 2, \cdots, 5$;

$\Phi_{ki}(x)$: the uncertain distribution of C_{ki}, $i = 1, 2, 3$, $k = 1, 2, \cdots, 5$;

C_{0i}: the uncertain inputs of the target DMU_0, $i = 1, 2, 3$;

$\Phi_{0i}(x)$: the uncertain distribution of C_{0i}, $i = 1, 2, 3$;

α_1: the requirement of learning difficulty;

α_2: the requirement of application difficulty;

α_3: the requirement of complexity;

A_{k1}: uncertain variable representing adaptability to mixed uncertainty, $k = 1, 2, \cdots, 5$;

A_{k2}: uncertain variable representing adaptability to separate uncertainty, $k = 1, 2, \cdots, 5$;

A_{k3}: uncertain variable representing adaptability to big data, $k = 1, 2, \cdots, 5$;

A_{k4}: uncertain variable representing adaptability to spare data, $k = 1, 2, \cdots, 5$;

A_{k5}: uncertain variable representing adaptability to duality axiom, $k = 1, 2, \cdots, 5$;

A_{k6}: uncertain variable representing adaptability to slow attenuation, $k = 1, 2, \cdots, 5$;

$\Psi_{kj}(x)$: the uncertain distribution A_{kj}, $j = 1, 2, \cdots, 6$, $k = 1, 2, \cdots, 5$;

A_{0j}: the uncertain outputs of the target DMU_0, $j = 1, 2, \cdots, 6$;

$\Psi_{0j}(x)$: the uncertain distribution of A_{0j}, $j = 1, 2, \cdots, 6$;

β_1: the requirement of adaptability to mixed uncertainty;

β_2: the requirement of adaptability to separate uncertainty;

β_3: the requirement of adaptability to big data;

β_4: the requirement of adaptability to spare data;

β_5: the requirement of adaptability to duality axiom;

β_6: the requirement of adaptability to slow attenuation.

The evaluating model is written as:

$$
\begin{aligned}
&\max \quad \theta = \sum_{i=1}^{3} s_i^- + \sum_{j=1}^{6} s_j^+ \\
&subject\ \ to: \\
&\begin{cases}
\mathcal{M}\left\{ \sum_{k=1}^{5} C_{kj}\lambda_k \le C_{0i} - s_i^- \right\} \ge 1-\alpha_i, & i = 1,\ 2, 3 \\
\mathcal{M}\left\{ \sum_{k=1}^{5} A_{kj}\lambda_k \ge A_{0j} + s_j^+ \right\} \ge 1-\beta_j, & j = 1,\ 2, \ldots 6 \\
\sum_{k=1}^{5} \lambda_k = 1 \\
\lambda_k \ge 0, \quad k = 1, 2, \cdots, 5 \\
s_i^- \ge 0, \quad i = 1, 2, 3 \\
s_j^+ \ge 0, \quad j = 1, 2, \ldots, 6,
\end{cases}
\end{aligned}
\tag{11}
$$

where s_i^- and s_j^+ represent input and output slacks, respectively.

In uncertainty theory, α_i and β_j are regarded as belief degrees, which measures the strength with which we believe the event will happen. In the proposed model, α_i and β_j are used to represent the requirement level from the demand side.

Theorem 10. (see Wen et al. [30]) Assume that $\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni}$ are independent uncertain inputs with regular uncertainty distributions $\Phi_{1i}, \Phi_{2i}, \cdots, \Phi_{ni}$ for each i, $i = 1, 2, \cdots, p$, and $\tilde{y}_{1i}, \tilde{y}_{2i}, \cdots, \tilde{y}_{ni}$ are independent uncertain outputs with uncertainty distribution $\Psi_{1i}, \Psi_{2i}, \cdots, \Psi_{ni}$ for each j, $j = 1, 2, \cdots, q$. Then

$$
\begin{aligned}
&\mathcal{M}\left\{ \sum_{k=1}^{n} \tilde{x}_{ki}\lambda_k = \tilde{x}_{0i} - s_i^- \right\} \ge \alpha, \; i = 1, 2, \ldots, p \\
&\mathcal{M}\left\{ \sum_{k=1}^{n} \tilde{y}_{kj}\lambda_k = \tilde{y}_{0j} + s_i^+ \right\} \ge \alpha, \; j = 12, \ldots, q
\end{aligned}
\tag{12}
$$

holds if and only if

$$
\begin{aligned}
&\sum_{k=1, k\ne 0}^{n} \lambda_k \Phi_{ki}^{-1}(\alpha) + \lambda_0 \Phi_{0i}^{-1}(1-\alpha) \le \lambda_0 \Phi_{0i}^{-1}(1-\alpha) - s_i^-, \quad i = 1, 2, \ldots, p \\
&\sum_{k=1, k\ne 0}^{n} \lambda_k \Psi_{kj}^{-1}(1-\alpha) + \lambda_0 \Psi_{0j}^{-1}(\alpha) \le \lambda_0 \Psi_{0j}^{-1}(\alpha) + s_j^+, \quad j = 1, 2, \ldots, q.
\end{aligned}
\tag{13}
$$

According to Theorem 10, the above model holds if and only if

$$
\begin{aligned}
&\max \quad \theta = \sum_{i=1}^{3} s_i^- + \sum_{j=1}^{6} s_j^+ \\
&subject\ \ to: \\
&\begin{cases}
\sum_{k=1, k\ne 0}^{5} \lambda_k \Phi_{ki}^{-1}(1-\alpha_i) + \lambda_0 \Phi_{0i}^{-1}(\alpha_i) \le \Phi_{0i}^{-1}(\alpha_i) - s_i^-, & i = 1, 2, 3 \\
\sum_{k=1, k\ne 0}^{5} \lambda_k \Psi_{kj}^{-1}(\beta_j) + \lambda_0 \Psi_{0j}^{-1}(1-\beta_j) \ge \Psi_{0j}^{-1}(1-\beta_j) + s_j^+, & j = 1, 2, \ldots, 6 \\
\sum_{k=1}^{5} \lambda_k = 1 \\
\lambda_k \ge 0, \quad k = 1, 2, \cdots, 5 \\
s_i^- \ge 0, \quad i = 1, 2, 3 \\
s_j^+ \ge 0, \quad j = 1, 2, \cdots, 6,
\end{cases}
\end{aligned}
\tag{14}
$$

Table 2
Scores of the outputs of belief reliability.

	10	30	70	90
A_1	0.02	0.31	0.44	0.74
A_2	0.14	0.15	0.27	0.85
A_3	0.07	0.33	0.46	0.77
A_4	0.40	0.59	0.59	0.85
A_5	0.03	0.06	0.59	0.93
A_6	0.29	0.49	0.60	0.92

Table 3
Parameters of input and output variables.

k	1		2		3		4		5	
	e	σ	e	σ	e	σ	e	σ	e	σ
C_{k1}	28.24	39.38	17.73	41.54	40.91	46.73	43.89	43.70	27.84	45.98
C_{k2}	39.38	39.33	32.26	39.31	42.80	36.40	40.62	42.72	39.44	43.10
C_{k3}	40.75	36.49	36.59	45.12	56.74	45.59	37.08	37.62	38.70	37.12
A_{k1}	79.66	59.94	47.31	60.54	67.68	93.95	79.41	63.92	68.16	49.82
A_{k2}	65.70	25.11	51.88	77.90	31.85	53.08	67.29	55.89	77.38	13.45
A_{k3}	55.64	52.87	65.01	89.80	78.65	156.82	72.89	64.32	64.23	52.27
A_{k4}	50.28	66.33	48.62	35.60	60.27	33.67	71.11	57.28	24.97	94.68
A_{k5}	22.15	72.26	76.29	15.40	71.49	42.41	191.42	129.56	65.17	20.39
A_{k6}	79.58	124.79	120.45	71.62	125.68	173.76	70.31	102.96	38.27	62.51

where s_i^- and s_j^+ represent input and output slacks, respectively.

Theorem 11. (see Wen [40]) The objective value in Eq. (14) is an increasing function of requirements α_i and β_j.

According to Theorem 11, we can find that the higher the requirements are, the bigger the optimal result θ^* is. In other words, the index with a higher requirement level is more important than the index with a lower requirement level. Therefore, the proposed model can be used to measure the different realities by adjusting the levels of requirement for different indicators (inputs and outputs).

Evaluating criteria: The smaller the optimal result θ^* is, the more appropriate the candidate metric is.

According to the evaluating criteria, the candidate metric with the minimal optimal result is regarded as the most appropriate choice. However, it is possible to get the minimum value for multiple optimal results at the same time. This is because the candidate metrics all have obvious advantages at the given requirement level. In that case, decision makers can make the final choice based on their preferences.

5. A numerical example

The proposed uncertain DEA model for uncertainty metrics is explained using a series of cases which are used to describe the effect of requirements and the process of decision-making.

Four experts were invited to participate in the process of expert elicitation. As an illustration, Table 2 shows the mean scores of belief reliability evaluated by four experts, which are used to approximate uncertainty distributions of the outputs of belief reliability.

According to the maximum entropy principle, we assume that the input and output variables follow normal uncertainty distributions as given in Eq. (5). The parameters of input and output variables are estimated by principle of least squares and the results are shown in Table 3.

There are four cases used to describe four different requirements. In case 1, adaptability to mixed uncertainty is less weighty. In case 2, adaptability to slow attenuation is less important. In case 3, learning difficulty has fewer restrictions. In case 4, application difficulty, adaptability to separate uncertainty and spare data are more important. Table 4 shows precise numerical description of the above requirements. Table 5 shows the final optimal results.

According to the evaluating criteria, the recommended metrics for case 1, 2 and 4 are EB reliability, FIB reliability and posbist reliability,

Table 4
Numerical description of the requirements.

	α_1	α_2	α_3	β_1	β_2	β_3	β_4	β_5	β_6
Case 1	0.9	0.9	0.9	0.5	0.9	0.9	0.9	0.9	0.9
Case 2	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.5
Case 3	0.5	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
Case 4	0.6	0.9	0.6	0.6	0.9	0.6	0.9	0.5	0.6

Table 5
Optimal results.

	EB reliability	IB reliability	FIB reliability	Posbist reliability	Belief reliability
Case 1	0.00	1261.86	1499.03	720.93	1293.40
Case 2	1240.81	1091.93	0.00	1096.07	1230.76
Case 3	1284.91	0.00	1585.87	1331.76	0.00
Case 4	673.53	607.51	666.94	0.00	696.04

respectively. Case 3 consists of two alternatives, IB reliability and belief reliability.

Case studies indicate that EB reliability is the most recommended choice when adaptability to mixed uncertainty is less important, FIB reliability is the most recommended choice when adaptability to slow attenuation is less important, IB reliability and belief reliability are the most recommended choices considering learning difficulty and posbist reliability is the most recommended choice considering application difficulty, adaptability to separate uncertainty and spare data.

From the case study, it is found that the proposed uncertainty optimal model is operative for both single requirement and multi-requirements. Further, it should be noted that there could be more than one recommended metrics according to the optimal results, in which case the final choice can depend on decision-maker preferences.

6. Conclusions

In this paper we have proposed an uncertain evaluating model based on data envelopment analysis for uncertainty metrics in reliability. From the view of theoretical completeness and engineering practicality, this paper has developed an evaluating index system including three capabilities and six adaptabilities. Since the indexes can only be measured by knowledge and experience of domain experts, a questionnaire survey process has been adopted to acquire empirical data. For the purpose of improving the effectiveness of decision-making, an uncertain optimal model has been developed with indicators from natural characteristics of metrics and the requirements from the demand side. The proposed model can be used to measure the different realities by adjusting the level of requirements and provide a new method in the selection among the uncertainty metrics in reliability.

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A Model-Based Reliability Metric Considering Aleatory and Epistemic Uncertainty

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ABSTRACT Model-based reliability analysis and assessment methods rely on models, which are assumed to be precise, to predict reliability. In practice, however, the precision of the model cannot be guaranteed due to the presence of epistemic uncertainty. In this paper, a new reliability metric, called belief reliability, is defined to explicitly account for epistemic uncertainty in model-based reliability analysis and assessment. A new method is developed to explicitly quantify epistemic uncertainty by measuring the effectiveness of the engineering analysis and assessment activities related to reliability. To evaluate belief reliability, an integrated framework is presented where the contributions of design margin, aleatory uncertainty, and epistemic uncertainty are integrated to yield a comprehensive and systematic description of reliability. The developed methods are demonstrated by two case studies.

INDEX TERMS Reliability, physics-of-failure, epistemic uncertainty, model uncertainty, belief reliability.

ACRONYMS

AUF	Aleatory Uncertainty Factor
ESV	Electrohydraulic Servo Valve
EU	Epistemic Uncertainty
EUF	Epistemic Uncertainty Factor
FMECA	Failure Mode, Effect and Criticality Analysis
FRACAS	Failure Report, Analysis, and Corrective Action System
HC	Hydraulic Cylinder
HSA	Hydraulic Servo Actuator
LTB	Larger-the-better
NTB	Nominal-the-better
RGT	Reliability Growth Test
RET	Reliability Enhancement Test
RST	Reliability Simulation Test
SBC	Single Board Computer
STB	Smaller-the-better

NOTATIONS

m	Performance margin
p	Performance parameter
p_{th}	Functional threshold
R_B	Belief reliability
R_p	Probabilistic reliability
m_d	Design margin
σ_m	Aleatory uncertainty factor
σ_e	Epistemic uncertainty factor
y	Effectiveness of the EU-related engineering activities

I. INTRODUCTION

Reliability refers to the ability of a component or system to perform a required function for a given period of time when used under stated operating conditions [1]. Traditionally, reliability is measured by the probability that functional failure does not occur in the considered period of time and failure data are used for its estimation based on statistical methods [2]. In practice, however, failure data are often scarce (if available at all), which defies the use of classical statistical methods and challenges Bayesian methods with respect to the assumption of subjective prior distributions [3]. Due to the problem of limited failure data, model-based methods (*cf.* physics-of-failure (PoF) methods [4], structural reliability methods [5], *etc.*) are widely applied to predict reliability, by deterministically describing the degradation and failure processes using deterministic failure behavior models. More specifically, it is assumed that:

1) the failure behavior of a component or a system can be described by a deterministic model;
2) random variations in the variables of the deterministic model are the sole source of uncertainty.

The probabilistic quantification of reliability is, then, obtained by propagating uncertainties through the model analytically or numerically, *e.g.*, by Monte Carlo simulation [6]–[8].

The random variations represent the uncertainty inherent in the physical behavior of the system and are referred to as aleatory uncertainty [9]. However, the model-based methods are also subject to epistemic uncertainty due to incomplete knowledge on the degradation and failure processes [10], [11]. According to Aven and Zio [12] and Bjerga *et al.* [13], epistemic uncertainty may arise because:

1) the deterministic model cannot exactly describe the failure process, *e.g.*, due to incomplete understanding of the failure causes and mechanisms (model uncertainty, also known as structural uncertainty);
2) the precise values of the model parameters might not be accurately estimated due to lack of data in the actual operational and environmental conditions (parameter uncertainty).

In this paper, we introduce a new reliability metric, belief reliability, to explicitly consider the effect of epistemic uncertainty on the model-based methods. For illustrative purposes, we consider only model uncertainty in this paper. However, the framework can be easily extended to deal with parameter uncertainty.

In literature, various approaches have been developed to consider model uncertainty. Mosleh and Droguett reviewed a number of approaches for model uncertainty assessment and compared them in terms of theoretical foundations and domains of application [14], [15]. Among them, the alternate hypotheses approach and the adjustment factor approach are two most widely applied ones [16]. The alternate hypotheses approach identifies a family of possible alternate models and probabilistically combines the predictions of them based on Bayesian model averaging, where the probability of each model is evaluated from experimental data or expert judgements [17], [18]. Apostolakis [19] addressed the issue of model uncertainty in probabilistic risk assessment using the alternate hypotheses approach. Park and Grandhi [20] quantified the model probability in the alternate hypotheses approach by the measured deviations between experimental data and model predictions. In [21], two crack models were probabilistically combined using the alternate hypotheses approach to estimate the failure probability of a butt weld. Other applications of the alternate hypotheses approach include sediment transport models [22], identification of benchmark doses [23], precipitation modeling [24], *etc*.

In the adjustment factor approach, the model uncertainty is addressed by modifying a benchmark model (the one that we have highest confidence in) with an adjustment factor, which is assumed to be uncertain, and is either added to or multiplied by the prediction results of the model [16], [25]. In [26], the adjustment factor approach was used to combine experts' estimates according to Bayes' theorem. Zio and Apostolakis [16] used the approach to assess the risk of radioactive waste repositories. Fischer and Grandhi [27] applied an adjustment factor to low-fidelities models so as to scale them to high-fidelity models. In a series of studies conducted in [25] and [28]–[30], the adjustment factor approach was combined with the alternate hypotheses approach by introducing an adjustment factor to quantify the uncertainty in each alternate model; the model uncertainty was, then, evaluated by averaging all the models according to the alternate hypotheses approach.

The alternate hypotheses approach requires enumerating a set of mutually exclusive and collectively exhaustive models [15]. In the case of model-based reliability methods, however, it is impossible for us to enumerate all the possible models, which limits the application of the alternate hypotheses approach. Hence, we adopt the adjustment factor approach in this paper to develop a new reliability metric to describe the effect of epistemic uncertainty (model uncertainty) on the model-based reliability methods.

In the adjustment factor approaches, epistemic uncertainty is quantified by the adjustment factor, which is often determined based on validation test data (for example, see [18] or [30]). In practice, however, due to limited time and resources, it is hard, if not impossible, to gather sufficient validation test data. Resorting to expert judgements might offer an alternative solution (for example, see [16]), but they could be criticized for being too subjective. On the other hand, epistemic uncertainty relates to the knowledge on the component or system functions and failure behaviors: as this knowledge is accumulated, epistemic uncertainty is reduced. In the life cycle of a component or system, the knowledge is gained by implementing a number of reliability analysis-related engineering activities, whose purpose is to help designers better understand potential failure modes and mechanisms. For example, through Failure Mode, Effect and Criticality Analysis (FMECA), potential failure modes and their effects could be identified, so that the designer can better understand the product's failure behaviors [31]. Similar engineering activities include Failure Report, Analysis, and Corrective Action System (FRACAS) [32], Reliability Growth Test (RGT) [33], Reliability Enhancement Test (RET) [32], Reliability Simulation Test (RST) [34], [35], *etc*. In this paper, we develop a new quantification method for the epistemic uncertainty in the adjustment factor method, based on the effectiveness of these engineering activities.

The contributions of this paper are summarized as follows:

1) a new reliability metric, the belief reliability, is developed to explicitly consider epistemic uncertainty in the model-based reliability methods;
2) a new method is developed to quantify epistemic uncertainty, based on the effectiveness of the engineering activities related to the reliability analysis and assessment of components and systems;
3) a method is developed to evaluate the belief reliability of components and systems, based on the integration of design margin, aleatory uncertainty and epistemic uncertainty.

The rest of the paper is organized as follows. In Section II, belief reliability is defined to account for the effect of epistemic uncertainty in model-based reliability methods. In Section III, epistemic uncertainty is quantified based on the effectiveness of the related engineering activities and a belief reliability evaluation method is developed. Section IV presents two case studies to demonstrate the developed methods. Finally, the paper is concluded in Section V with a discussion on future works.

II. DEFINITION OF BELIEF RELIABILITY

In this section, we introduce a new metric of reliability, belief reliability, to explicitly account for the influence of epistemic uncertainty on model-based reliability methods. We start with a brief introduction of the model-based reliability method in subsection II-A. Then, belief reliability is defined in subsection II-B.

A. MODEL-BASED RELIABILITY METHODS

For a general description of model-based reliability methods, we introduce the concepts of performance parameter and performance margin:

Definition 1 (Performance Parameter): Suppose failure occurs when a parameter p reaches a threshold value p_{th}. Then, the parameter p is referred to as a performance parameter, while the threshold value p_{th} is referred to as the functional failure threshold associated with p.

According to *Definition* 1, performance parameters and functional failure thresholds define the functional requirements on a system or a component, for which three categories exist in practice:

1) Smaller-the-better (STB) parameters: if failure occurs when $p \geq p_{th}$, then, the performance parameter p is a STB parameter.
2) Larger-the-better (LTB) parameters: if failure occurs when $p \leq p_{th}$, then, the performance parameter p is a LTB parameter.
3) Nominal-the-better (NTB) parameters: if failure occurs when $p \leq p_{th,L}$ or $p \geq p_{th,U}$, then, the performance parameter p is a NTB parameter.

Definition 2 (Performance Margin): Suppose p is a performance parameter and p_{th} is its associated functional failure threshold; then,

$$m = \begin{cases} \dfrac{p_{th} - p}{p_{th}}, & \text{if } p \text{ is STB,} \\ \dfrac{p - p_{th}}{p_{th}}, & \text{if } p \text{ is LTB,} \\ \min\left(\dfrac{p_{th,U} - p}{p_{th,U}}, \dfrac{p - p_{th,L}}{p_{th,L}}\right), & \text{if } p \text{ is NTB} \end{cases} \quad (1)$$

is defined as the (relative) performance margin associated with the performance parameter p.

Remark 1: From Definition 2, performance margin is a unitless quantity and failure occurs whenever $m \leq 0$.

In the model-based reliability methods, it is assumed that the performance margin can be described by a deterministic model, which is derived based on knowledge of the functional principles and failure mechanisms of the component [5], [36]. Conceptually, we assume that the performance margin model has the form

$$m = g_m(\mathbf{x}), \quad (2)$$

where $g_m(\cdot)$ denotes the deterministic model which predicts the performance margin and $\mathbf{x}$ is a vector of input variables.

In the design and manufacturing processes of a product, there are many uncertain factors influencing the input $\mathbf{x}$ of (2). Thus, the values of $\mathbf{x}$ may vary from product to product of the same type. Usually, this product-to-product variability is described by assuming that $\mathbf{x}$ is a vector of random variables with given probability density functions. Then, m is also a random variable and reliability R_p is defined as the probability that m is greater than zero. The subscript p is used to indicate that R_p is a probability measure. Given the probability density function of $\mathbf{x}$, denoted by $f_X(\cdot)$, R_p can be calculated by:

$$R_p = Pr\left(g_m(\mathbf{x}) > 0\right) = \int \cdots \int_{g_m(\mathbf{x})>0} f_X(\mathbf{x})\mathrm{d}\mathbf{x}. \quad (3)$$

B. DEFINITION OF BELIEF RELIABILITY

Belief reliability is defined in this subsection to explicitly account for the effect of epistemic uncertainty in model-based reliability methods. For this, we first define design margin and Aleatory Uncertainty Factor (AUF):

Definition 3 (Design Margin): Suppose the performance margin of a component or a system can be calculated by (2). Then, design margin m_d is defined as

$$m_d = g_m(\mathbf{x}_N), \quad (4)$$

where $\mathbf{x}_N$ is the nominal values of the parameters.

Definition 4 (Aleatory Uncertainty Factor (AUF)): Suppose R_p is the probabilistic reliability calculated from the performance margin model using (3). Then, AUF σ_m is defined as

$$\sigma_m = \frac{m_d}{Z_{R_p}}, \quad (5)$$

where Z_{R_p} is the value of the inverse cumulative distribution function of a standard normal distribution evaluated at R_p.

Further, let equivalent design margin M_E to be

$$M_E = m_d + \epsilon_m, \quad (6)$$

where $\epsilon_m \sim \text{Normal}(0, \sigma_m^2)$. It is easy to verify that $M_E \sim \text{Normal}(m_d, \sigma_m^2)$ and R_p can be calculated as the probability that $M_E > 0$, as shown in Figure 1 (a). Therefore, the probabilistic reliability can be quantified by the equivalent performance margin and further by m_d and σ_m, where

- m_d describes the inherent reliability of the product when all the input variables take their nominal values. Graphically, it measures the distance from the center of the equivalent performance margin distribution to the boundaries of the failure region, as shown in Figure 1 (a);

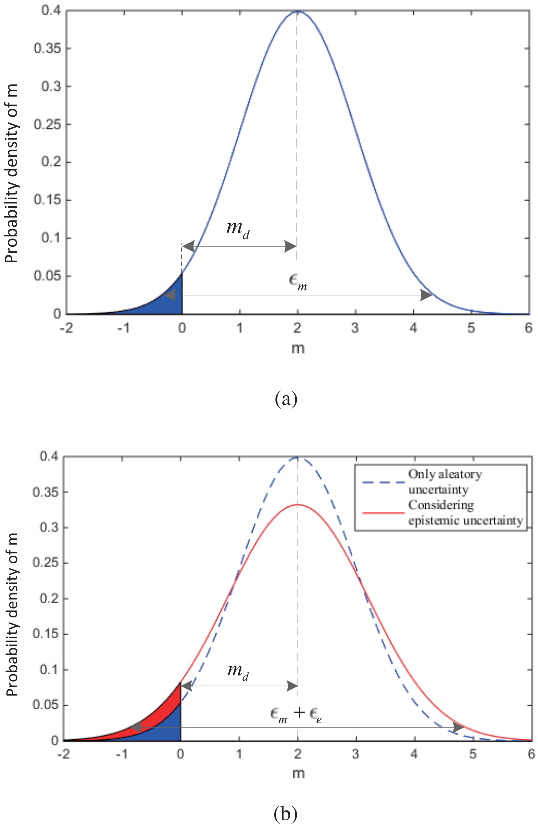

FIGURE 1. Epistemic uncertainty effect on the distribution of the equivalent performance margin. (a) Aleatory distribution. (b) Effect of epistemic uncertainty.

- σ_m accounts for the uncertainty resulting from the product-to-product random variations, *e.g.*, the tolerance of manufacturing processes, the variability in material properties, *etc*. Usually, these random variations are controlled by engineering activities such as tolerance design, environmental stress screening, stochastic process control, *etc* [11].

To further account for the effect of epistemic uncertainty, it is assumed that:

$$M_E = m_d + \epsilon_m + \epsilon_e, \tag{7}$$

where ϵ_e is an adjustment factor [16] and $\epsilon_e \sim \text{Normal}(0, \sigma_e^2)$. Parameter σ_e is defined as Epistemic Uncertainty Factor (EUF) and it quantifies the effect of epistemic uncertainty. The physical meaning of (7) is explained in Figure 1 (b): epistemic uncertainty introduces additional dispersion to the aleatory distribution of the equivalent performance margin. The degree of the dispersion is related to the knowledge we have on the failure process of the product, *i.e.*, the more knowledge we have, the less value σ_e takes.

Considering the assumption made in (7), we can, then, define the belief reliability as follows:

Definition 5 (Belief Reliability): The reliability metric

$$R_B = \Phi_N\left(\frac{m_d}{\sqrt{\sigma_m^2 + \sigma_e^2}}\right) \tag{8}$$

is defined as belief reliability, where $\Phi_N(\cdot)$ is the cumulative distribution function of a standard normal random variable.

Belief reliability can be interpreted as our belief degree on the product reliability, based on the knowledge of design margin, aleatory uncertainty and epistemic uncertainty. In the following, we discuss respectively how design margin, aleatory uncertainty and epistemic uncertainty influence the value of belief reliability.

Discussion 1: It is obvious from (8) that $R_B \in [0, 1]$, where

- $R_B = 0$ indicates that we believe for sure that a component or system is unreliable, *i.e.*, it cannot perform its desired function under stated time period and operated conditions.
- $R_B = 1$ indicates that we believe for sure that a component or system is reliable, *i.e.*, it can perform its desired function under stated time period and operated conditions.
- $R_B = 0.5$ indicates that we are most uncertain about the reliability of the component or system [37].
- $R_{B,A} > R_{B,B}$ indicates that we believe that product A is more reliable than product B.

Discussion 2 (Variation of R_B With the Design Margin): From (8), it is easy to see that R_B is an increasing function of m_d, as illustrated by Figure 2, which is in accordance with the intuitive fact that when the design margin is increased, the component or system becomes more reliable.

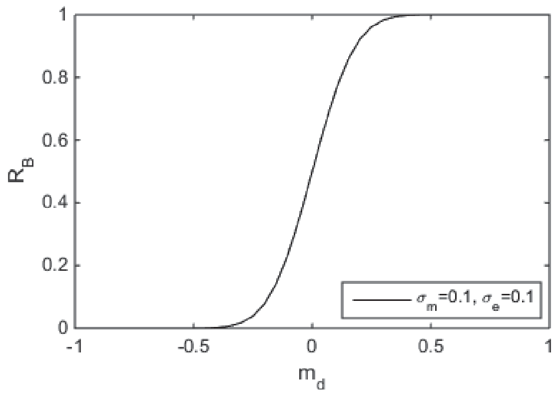

FIGURE 2. Influence of m_d on R_B.

Besides, it can be verified from (8) that if $m_d = 0$, $R_B = 0.5$. This is because when $m_d = 0$, the product is at borderline between working and failure. Therefore, we are most uncertain about its reliability (For details, please refer to the maximum uncertainty principle in [37]).

Discussion 3 (Variation of R_B With the Aleatory Uncertainty): In (8), the effect of aleatory uncertainty is measured by the AUF, σ_m. Figure 3 shows the variation of R_B with σ_m, when σ_e is fixed, for different values of m_d. It can be seen from Figure 3 that when m_d and σ_e are fixed, R_B approaches 0.5 as σ_m increases to infinity. The result is easy to understand, since $\sigma_m \to \infty$ indicates the fact that uncertainty has the greatest influence.

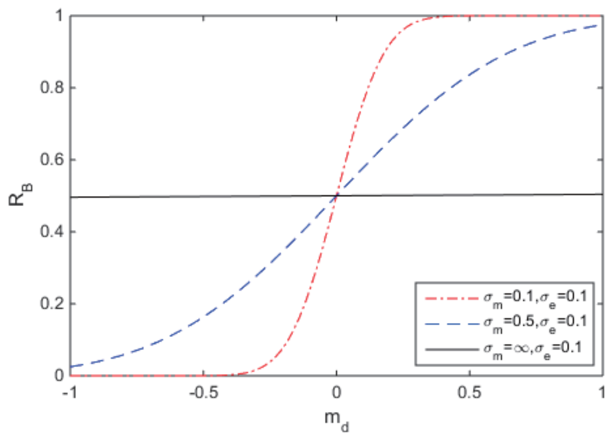

FIGURE 3. Variation of R_B with σ_m.

Discussion 4 (Variation of R_B With the Epistemic Uncertainty): In (8), the effect of epistemic uncertainty is measured by the EUF, σ_e. The variation of R_B with respect to σ_e is illustrated in Figure 4, with σ_m fixed to 0.2. From Figure 4, we can see that when $\sigma_e \rightarrow \infty$, R_B also approaches 0.5, for the same reason as the AUF.

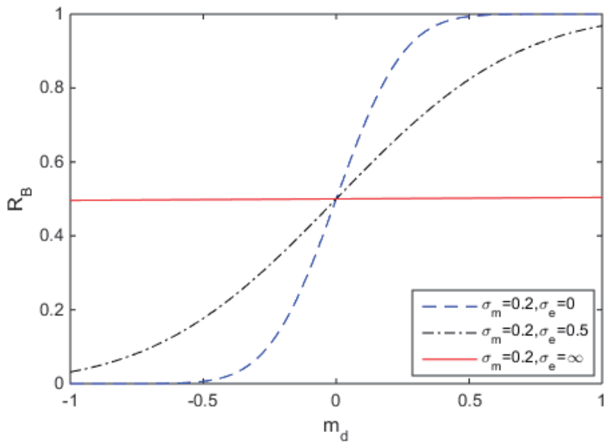

FIGURE 4. Variation of R_B with σ_e.

Besides, it can be shown from (8) and assumption (3) that as $\sigma_e \rightarrow 0$, R_B approaches the R_p calculated by the model-based reliability methods using equation (3). This is a natural result since $\sigma_e = 0$ is the ideal case for which there is no epistemic uncertainty, so that the product failure behavior is accurately predicted by the deterministic performance margin model and the aleatory uncertainty.

In practice, we always have $m_d \geq 0$ and $\sigma_e > 0$. Therefore,

$$R_B \leq R_p \tag{9}$$

where R_p is the probabilistic reliability predicted by (3) under the same conditions. Equation (9) shows that using belief reliability yields a more conservative evaluation result than using the probabilistic reliability, because belief reliability considers the effect of insufficient knowledge on the reliability evaluations.

III. EVALUATION OF BELIEF RELIABILITY

In this section, we discuss how to evaluate the belief reliability for a given product. A general framework for belief reliability evaluation is first given in subsection III-A. Then, a method is presented for evaluating epistemic uncertainty and determining the value of the EUF.

A. BELIEF RELIABILITY EVALUATION

The R_B defined in (8) incorporates the contributions of design margin m_d, aleatory uncertainty (represented by σ_m) and epistemic uncertainty (represented by σ_e). The contributions from the three factors should be evaluated individually and then, combined to evaluate the belief reliability of a component. Detailed procedures are presented in Figure 5.

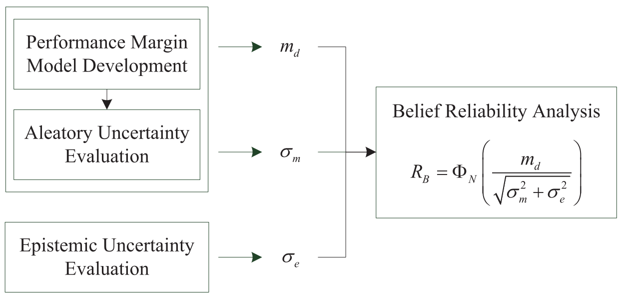

FIGURE 5. Procedures for component belief reliability evaluation.

Four steps comprise the evaluation procedure:

1) PERFORMANCE MARGIN MODEL DEVELOPMENT

First, a deterministic performance margin model is developed to predict the value of the performance margin m. The performance margin model can be developed based on knowledge of underlying functional principles and physics of failures. For a detailed discussion on how to develop performance margin models, readers might refer to [38] and [39].

2) ALEATORY UNCERTAINTY EVALUATION

Next, the values of m_d and σ_m are determined. The value of m_d is calculated based on (4), where all the input parameters of the performance margin model take their nominal values. To calculated the value of σ_m, the probabilistic reliability R_p is calculated first by propagating aleatory uncertainty in the model parameters according to (3). Either structural reliability methods [5] or Monte Carlo simulations [7] might be used for the calculation. Then, σ_m can be calculated by combining m_d and R_p using (5).

3) EPISTEMIC UNCERTAINTY EVALUATION

The value of σ_e is, then, determined by evaluating the effect and potential impact of epistemic uncertainty. In practice, epistemic uncertainty relates to the knowledge on the component or system functions and failure behaviors: as this

TABLE 1. Examples of EU-related engineering activities.

Activities	Contributions to gaining knowledge and reducing epistemic uncertainty
FMECA	FMECA helps designers to identify potential failure modes and understand their effects, so as to increase the designer's knowledge about potential failures [31].
FRACAS	By implementing FRACAS, knowledge on potential failure modes and mechanisms is accumulated based on previously ocurred failures and corrective actions [32].
RGT	In a RGT, cycles of Test Analysis and Fix (TAAF) are repeated until the product reaches its reliability requirements. In this way, designers' knowledge on the failure modes and mechanisms is accumulated [33].
RET	As the RGT, RET reduces epistemic uncertainty by stimulating potential failures, but using highly accelerated stresses, which can generate failures that are hard to be identified by analyses or conventional tests [32].
RST	In a RST, simulation tests are conducted based on physics-of-failure models to identify weak design points for the products. Knowledge of potential failure modes can be accumulated in this way [34, 35].

knowledge is accumulated, epistemic uncertainty is reduced. Hence, in this paper, we relate epistemic uncertainty to our state of knowledge on the product and its failure process and assess the value of σ_e based on the effectiveness of engineering activities that generate our knowledge base. Details on how to evaluate the value of σ_e is given in Section III-B.

4) BELIEF RELIABILITY EVALUATION

Following steps 1) - 3), the values of m_d, σ_m and σ_e are determined. Then, the belief reliability can be evaluated according to (8).

B. QUANTIFICATION OF EPISTEMIC UNCERTAINTY

In this section, we develop a method to quantify epistemic uncertainty based on the state of knowledge. In subsection III-B1, we discuss how to evaluate the state of knowledge, and then, in subsection III-B2, we quantify the effect of epistemic uncertainty in terms of σ_e.

1) EVALUATION OF THE STATE OF KNOWLEDGE

In the life cycle of a component or system, the knowledge on the products' failure behavior is gained by implementing a number of engineering activities of reliability analysis, whose purposes are to help designers better understand potential failure modes and mechanisms. In this paper, we refer to these engineering activities as epistemic uncertainty-related (EU-related) engineering activities. Table 1 lists some commonly encountered EU-related engineering activities and discusses their contributions to gaining knowledge and reducing epistemic uncertainty, where FMECA stands for Failure Mode, Effect and Criticality Analysis, FRACAS stands for Failure Reporting, Analysis, and Corrective Action System, RET stands for Reliability Enhancement Test, RGT stands for Reliability Growth Test and RST stands for Reliability Simulation Test.

In this paper, we make an assumption that the state of knowledge is directly related to the effectiveness of the EU-related engineering activities. Suppose there are n EU-related engineering activities in a product life cycle. Let $y_i, i = 1, 2, \cdots, n$ denote the effectiveness of the EU-related engineering activities, where $y_i \in [0, 1]$; the more effective the engineering activity is, the larger value the corresponding y_i takes. The values of y_i are determined by asking experts to evaluate the effectiveness of the EU-related engineering activities, based on a set of predefined evaluation criteria.

For example, the effectiveness of FMECA can be evaluated based on eight elements, as shown in Table 2. For each element, experts are invited to evaluate their performances according to the criteria listed in Table 2. Based on the evaluated performance, a score can be assigned to each element, denoted by $S_1, S_2, \cdots, S_8$. Then, the effectiveness of FMECA, denoted by y_1, can be determined by

$$y_1 = \frac{1}{8}\sum_{i=1}^{8} S_i. \tag{10}$$

The effectiveness of other EU-related engineering activities can be evaluated in a similar way, so that the values for $y_1, y_2, \cdots, y_n$ can be determined. Then, the state of knowledge about the potential failures of the component or system can be evaluated as the weighted average of $y_i, i = 1, 2, \cdots, n$:

$$y = \sum_{i=1}^{n} \omega_i y_i, \tag{11}$$

where ω_i is the relative importance of the ith engineering activity for the characterization of the potential failure behaviors, where $\sum_{i=1}^{n} \omega_i = 1$.

2) DETERMINATION OF EUF

Having determined the value of y, we need to define a function $\sigma_e = h(y)$, through which σ_e is determined. Since σ_e is a measure of the severity of epistemic uncertainty and y measures the state of knowledge, σ_e is negatively dependent on y. Theoretically, any monotonic decreasing function of y could serve as $h(y)$. In practice, the form of $h(y)$ reflects the decision maker attitude towards epistemic uncertainty and is related to the complexity of the product. Therefore, we propose $h(y)$ to be

$$h(y) = \begin{cases} \dfrac{1}{3\sqrt{y}} \cdot m_d, & \text{for simple products;} \\ \dfrac{1}{3y^6} \cdot m_d, & \text{for complex products;} \\ \dfrac{1}{3y^2} \cdot m_d, & \text{for medium complex products.} \end{cases} \tag{12}$$

TABLE 2. Evaluation criteria for FMECA.

Notations	Elements	Criteria	Scores
S_1	Definition of failures	The failure definition is clear and unambiguous, so that a complete analysis of failure modes could be conducted.	$S_1 = 3$
		The failure definition is defined, but unclear or ambiguous.	$S_1 = 1$
		The failures are undefined.	$S_1 = 0$
S_2	Coverage of failure modes	The considered failure modes cover all that have occurred historically in the products with similar functions, use and environmental conditions.	$S_2 = 3$
		A few uncritical failure modes are not considered in the analysis.	$S_2 = 1$
		A lot of critical failure modes are not considered in the analysis.	$S_2 = 0$
S_3	Completeness of failure mode analysis	The considered failure modes include both catastrophic failures as well as degradation failures.	$S_3 = 3$
		Only one of the two types is considered.	$S_3 = 1$
		None of the two types is considered.	$S_3 = 0$
S_4	Credibility of information sources	The considered failure modes come from real historical data of the product or similar products.	$S_4 = 3$
		The considered failure modes come from literature.	$S_4 = 1$
		The considered failure modes come from expert judgements.	$S_4 = 0$
S_5	Completeness of failure cause analysis	The analysis takes into account all the possible failure causes.	$S_5 = 3$
		A few noncritical failure causes are not considered in the analysis.	$S_5 = 1$
		A lot of critical failure causes are not considered in the analysis.	$S_5 = 0$
S_6	Completeness of failure effect analysis	Failure effects in both component and system levels are analyzed.	$S_6 = 3$
		Only one of the two levels is considered.	$S_6 = 1$
		Failure effects are not considered in the analysis.	$S_6 = 0$
S_7	Credibility of data sources	Criticality analysis is based on field data.	$S_7 = 3$
		Criticality analysis is based on data from literature.	$S_7 = 1$
		Criticality analysis is based on data from expert judgements.	$S_7 = 0$
S_8	Effectiveness of design improvements	Over 90% of the exposed design weaknesses are modified.	$S_8 = 3$
		The modified design weaknesses are between 60%-90%.	$S_8 = 1$
		Less than 60% of the exposed design weaknesses are modified.	$S_8 = 0$

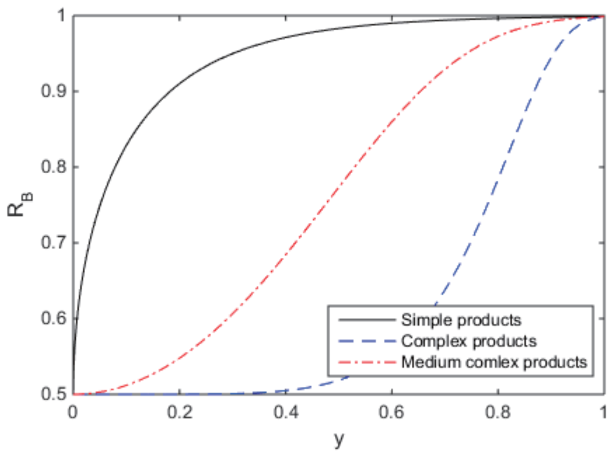

FIGURE 6. Different attitudes of the decision maker towards epistemic uncertainty.

By letting $\sigma_m = 0$ and m_d fixed to a constant value, the attitudes of the decision maker for different products can be investigated (see Figure 6):

- for simple products, R_B is a convex function of y, indicating that even when y is small, we can gather enough knowledge on the product function and failure behaviors, so that we can assign a high value to the belief reliability;
- for complex products, R_B is a concave function of y, indicating that only when y is large we can gather sufficient knowledge on the product function and failure behaviors, so that we can assign a high value to the belief reliability;
- the $h(y)$ for medium complex products lies between the two extremes.

IV. CASE STUDIES

In this section, we apply the developed belief reliability to evaluate the reliability of two engineering components/systems, *i.e.,* a Hydraulic Servo Actuator (HSA) in Section IV-A and a Single Board Computer (SBC) in Section IV-B. A comparison is also made on both cases with respect to the traditional probabilistic reliability metrics.

A. HYDRAULIC SERVO ACTUATOR (HSA)

The HSA considered in this paper comprises the six components, as listed in Table 3. The schematic of the HSA is given in Figure 7.

The required function of the HSA is to transform input electrical signals, x_{input}, into the displacement of the hydraulic cylinder (HC). The performance parameter of the HSA is the attenuation ratio measured in dB:

$$p_{\text{HSA}} = -20\lg\frac{A_{\text{HC}}}{A_{\text{obj}}}, \quad (13)$$

where, A_{HC} denotes the amplitude of the HC displacements when input signal x_{input} is a sinusoidal signal, and A_{obj} is the objective value of A_{HC}. Failure occurs when $p_{\text{HSA}} \geq p_{th} = 3$(dB). The belief reliability of the HSA is evaluated following the procedures in Figure 5.

TABLE 3. Components and tolerances of the HSA.

Component	ESV	Spool 1	Spool 2	Spool 3	Spool 4	HC
Parameters	CoD x_1	CoD x_2	CoD x_3	CoD x_4	CoD x_5	CoD x_6
Tolerances ($\times 10^{-3}$mm)	1 ± 0.015	7 ± 0.15	7 ± 0.15	7 ± 0.15	7 ± 0.15	10 ± 1.5
Distributions	$\mathrm{N}(1, 0.005^2)$	$\mathrm{N}(7, 0.05^2)$	$\mathrm{N}(7, 0.05^2)$	$\mathrm{N}(7, 0.05^2)$	$\mathrm{N}(7, 0.05^2)$	$\mathrm{N}(10, 0.5^2)$

ESV: Electrohydraulic servo valve
HC: Hydraulic cylinder

1) PERFORMANCE MARGIN MODEL DEVELOPMENT

The performance margin model is developed in two steps. First, a model for the p_{HSA} is developed based on hydraulic principles, with the help of commercial software AMESim. The AMESim model is given in Figure 7. Coherently with (2), the model in Figure 7 is written as

$$p_{\mathrm{HSA}} = g_{\mathrm{HSA}}(\mathbf{x}_{\mathrm{HSA}}). \tag{14}$$

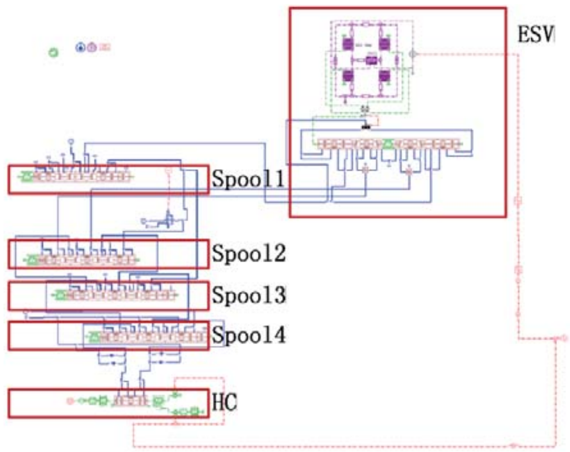

FIGURE 7. Schematic of the AMESim model to predict p_{HSA}.

Second, as p_{HSA} is a STB performance parameter, the performance margin of the HSA can be determined according to (1):

$$m_{\mathrm{HSA}} = \frac{1}{p_{th}} \left(p_{th} - g_{\mathrm{HSA}}(\mathbf{x}_{\mathrm{HSA}})\right). \tag{15}$$

2) ALEATORY UNCERTAINTY EVALUATION

The $\mathbf{x}_{\mathrm{HSA}}$ comprises six parameters, namely, the clearances on diameters (CoDs) of the six components of the HSA. The CoDs are subject to aleatory uncertainties from production and manufacturing processes, which are quantified by the tolerances in Table 3. For simplicity of illustration, it is assumed that all the six parameters follow normal distributions. Following the '3σ' principle (for references, see [40]), the probability density function for each parameter is determined and given in Table 3. The value of m_d is calculated by (4), where the nominal values are given in Table 3. The resulting m_d is 0.6928 (dB). The values of σ_m is determined using Monte Carlo simulations with a sample size $N = 3000$. The resulting σ_m is 0.0353 (dB).

3) EPISTEMIC UNCERTAINTY EVALUATION

Then, we need to determine the value of σ_e. In the development of the HSA, five EU-related engineering activities, i.e., FMECA, FRACAS, RGT, RET and RST have been conducted. Let y_i, $i = 1, 2, \cdots, 5$ denote the five engineering activities, respectively. The values of y_is can be determined by evaluating the effectiveness of these engineering activities, based on the procedures illustrated in Section III-B1. The result is $y_1 = 0.70$, $y_2 = 0.90$, $y_3 = 0.80$, $y_4 = 0.85$, $y_5 = 0.70$. In this case study, the engineering activities are assumed to have equal weights, $\omega_1 = \omega_2 = \cdots = \omega_5 = 1/5$, and then, according to (11), $y = 0.79$. Since the HSA has medium complexity, according to (12),

$$\sigma_e = \frac{1}{3y^2} \cdot m_d = 0.3700. \tag{16}$$

4) BELIEF RELIABILITY EVALUATION

Finally, the belief reliability can be predicted using (8) and the result is shown in Table 4. If we only consider the aleatory uncertainty, probabilistic reliability can be predicted using (3), whose value is also presented in Table 4 for comparisons. The result shows that, as expected, epistemic uncertainty reduces our confidence that the product will perform its function as designed, whereas probabilistic reliability would lead to overconfidence.

TABLE 4. Comparison between probabilistic reliability and belief reliability.

Types of reliability measures	Results
Probabilistic reliability calculated by (3)	0.9999
Belief reliability calculated by (8)	0.9688

Another major difference between belief reliability and probabilistic reliability is that belief reliability allows for the consideration of EU-related engineering activities in the reliability assessment, which are neglected in the probability-based reliability evaluation. For example, if the effectiveness of the EU-related engineering activities is increased from $y_1 = 0.70$, $y_2 = 0.90$, $y_3 = 0.80$, $y_4 = 0.85$, $y_5 = 0.70$ to $y_1 = y_2 = \cdots = y_5 = 0.9$, then, the belief reliability will increase from $R_{B,0} = 0.9688$ to $R_{B,1} = 0.9921$. In other words, in order to enhance the belief reliability, one not only needs to increase the design margin and reduce aleatory uncertainty by design, but also needs to reduce epistemic uncertainty by improving the state of knowledge, whereas probabilistic reliability focuses only on the former two aspects.

B. SINGLE BOARD COMPUTER

A SBC, as shown in Figure 8 [41], is chosen to demonstrate the time-dependent belief reliability analysis for electrical systems.

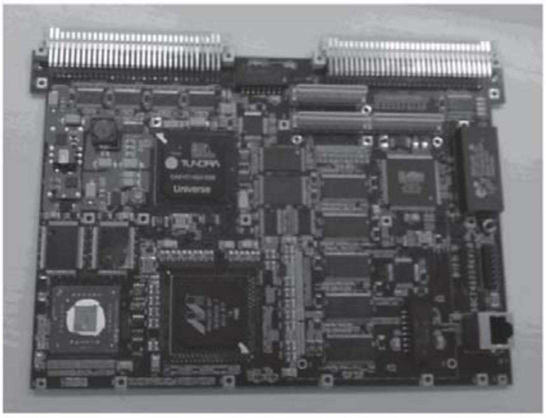

FIGURE 8. A SBC [41].

TABLE 5. Predicted failure rates of the SBC [41].

Components	Number	Predicted failure rate ($\times10^{-9}$ (h^{-1}))
IC	51	384.8
Crystal oscillator	4	56
Inductance	6	6.56
Connector	9	32.76
Capacitor	631	40.60
Resistance	545	648.91
Others	20	16.16
Total	1266	1186

A probabilistic reliability analysis was conducted in [41] based on the parts-counting reliability prediction method in [42]. The times to failure of both the components are assumed to be exponentially distributed and their failure rates are predicted based on the database in [42], as shown in Table 5. The failure rate of the SBC can, then, be calculated by summing over all the components' failure rates. Hence, the predicted probabilistic reliability is

$$R_p(t) = \exp\{-1.186 \times 10^{-6}t\}, \tag{17}$$

where the unit of t is hour.

The probabilistic reliability in (17) is a time-dependent function. To further evaluate the belief reliability, first note by substituting (5) into (8), we have

$$R_B = \frac{1}{\sqrt{\left(\frac{1}{Z_{R_p}}\right)^2 + \left(\frac{\sigma_e}{m_d}\right)^2}}. \tag{18}$$

Since R_p is time-dependent, the belief reliability is also a time-dependent function and can be calculated by using (18) recursively at each time t:

$$R_B(t) = \frac{1}{\sqrt{\left(\frac{1}{Z_{R_p(t)}}\right)^2 + \left(\frac{\sigma_e}{m_d}\right)^2}}, \tag{19}$$

where $R_p(t)$ is the time-dependent probabilistic reliability and σ_e is the EUF evaluated using the procedures in Section III-B.

The effectiveness of the five EU-related engineering activities, i.e., FMECA, FRACAS, RGT, RET and RST, can be assessed using the procedures illustrated in Section III-B1: $y_1 = 0.60$, $y_2 = 0.80$, $y_3 = 0.70$, $y_4 = 0.75$, $y_5 = 0.55$. As the previous case study, we also assume that the five activities have equal weights. From (11), $y = 0.68$. By assessing the configuration of the SBC, it is determined that it has medium complexity. Therefore, by substituting (12) and (17) into (19), the belief reliability of the SBC can be calculated, as shown in Figure 9.

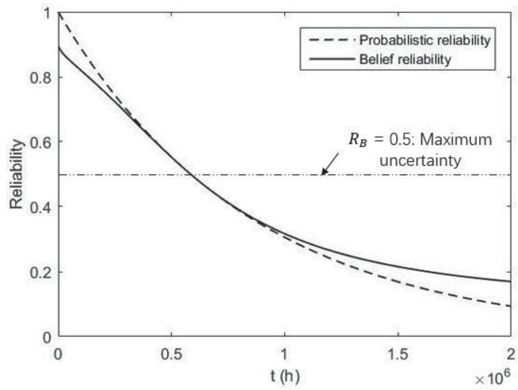

FIGURE 9. Belief reliability of the SBC.

It can be seen from Figure 9 that the belief reliability curve is more close to $R_B = 0.5$ than the probabilistic reliability. This is because $R_B = 0.5$ corresponds to the state of maximum uncertainty, since we cannot differentiate whether the system is more likely to be working or failure (for details, please refer to maximum uncertainty principle in [37]). Since belief reliability considers the influence of epistemic uncertainty, it yields a more uncertain result than the probabilistic reliability.

A sensitivity analysis is conducted with respect to y to further investigate the influence of epistemic uncertainty on belief reliability. The results are given in Figure 10. It can be seen from Figure 10 that the value of y significantly impacts R_B: a larger value of y, which indicates improvements on the effectiveness of the EU-related engineering activities, tends to make the belief reliability moving towards the probabilistic reliability; while a lower value of y tends to make the belief reliability moving towards 0.5, which is the state of maximum uncertainty. This demonstrates that, compared to the traditional probabilistic reliability, belief reliability allows for the explicit consideration of epistemic uncertainty and

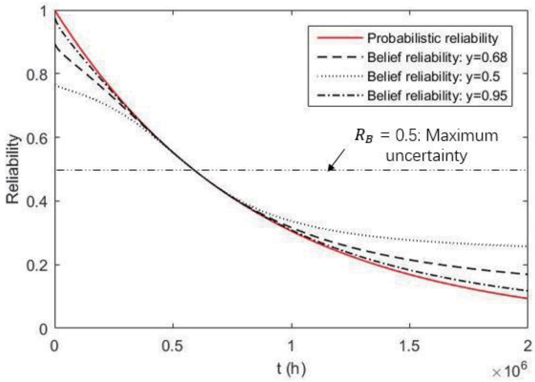

FIGURE 10. Belief reliability of the SBC.

EU-related engineering activities in the reliability assessment. In other words, in order to enhance the belief reliability, one not only needs to increase the design margin and reduce aleatory uncertainty by design, but also needs to reduce epistemic uncertainty by improving the state of knowledge.

V. CONCLUSION

In this paper, a new metric of belief reliability has been introduced to explicitly incorporate the influence of epistemic uncertainty into model-based methods of reliability assessments. To quantify the effect of epistemic uncertainty, an evaluation method is proposed, based on the effectiveness of engineering activities related to reliability analysis and assessment. The proposed belief reliability evaluation method integrates design margin, aleatory uncertainty and epistemic uncertainty for a comprehensive and systematic characterization of reliability. Two numerical case studies demonstrate the benefits of belief reliability compared to the traditional probability-based reliability metrics, with the explicit consideration of epistemic uncertainty.

Compared to the traditional probabilistic reliability metrics, belief reliability explicitly considers the effect of epistemic uncertainty and allows considering EU-related engineering activities in reliability assessment. We believe that as a new reliability metric, belief reliability is beneficial in reliability engineering practices, since epistemic uncertainty is a severe problem for real-world products, especially for those in design and development phases. An interesting future work is to define a mathematical theory to model belief reliability and its time-dependence. Various mathematical theories dealing with epistemic uncertainty can be considered, *e.g.*, Bayesian theory, evidence theory, possibility theory, uncertainty theory, *etc*. Besides, methods of scoring the effectiveness of engineering activities should be further investigated.

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Uncertain optimization model for multi-echelon spare parts supply system

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ABSTRACT

The optimization of spare parts inventory for equipment system is becoming a dominant support strategy, especially in the defense industry. Tremendous researches have been made to achieve optimal support performance of the supply system. However, the lack of statistical data brings limitations to these optimization models which are based on probability theory. In this paper, personal belief degree is adopted to compensate the data deficiency, and the uncertainty theory is employed to characterize uncertainty arising from subjective personal cognition. A base-depot support system is taken into consideration in the presence of uncertainty, supplying repairable spare parts for equipment system. With some constraints such as costs and supply availability, the minimal expected backorder model and the minimal backorder rate model will be presented based on uncertain measure. Genetic algorithm is adopted in this paper to search for optimal solution. Finally, a numerical example is employed to illustrate the feasibility of the optimization models.

1. Introduction

Inventory management has always been a major issue since last century. Some factors, such as the huge technological leap, the complexity of product systems, the expansion of outsourcing and supply networks, have made the system performance increasingly dependent on supply capacity and jointly promoted the complicacy of inventory management (Wagner and Neshat [29]).

Different from the general inventory management applied in civil use, inventory system of military equipment has its own characteristics: Spare parts inventory system supplies replacements for equipment system made of tens of thousands of items. Thus the strategy for inventory management depends heavily on the performance and maintenance of the system. What is more important, backorder in a military equipment system will bring sequential immeasurable damages which are more than those brought by the falling infinite economic loss. Focused on spare parts order strategy, inventory theory becomes more vital for a military equipment system to solve the issue about how to set and maintain a moderate stock level to strike a balance between the loss of downtime caused by under stock and the holding cost for excessive stock [28].

The development of inventory management has witnessed the substantial evolutionary process fulfilled with endeavors of enormous experts and scholars. In 1913, Harris [8] first studied the optimal economical solution to inventory problem and proposed the EOQ model, which has laid the foundation for later deterministic inventory models. Then exponential smoothing based on historical data was proposed by Syntetos *et al.* [26] to forecast consecutive demand. With this method, the demand of next period was obtained by smoothing the contemporary demand, considering that later data is more influential on future demand. Bootstrap method [5], which sampled the historical data to produce virtual data for demand forecasting, had a wide application.

However, the instability and supply uncertainty of inventory system have increased, which makes it more complicated for rational allocation of spare parts and appeals to many experts for exploration. Large numbers of researches have been conducted according to the probability theory, operational research and mathematical statistics in stochastic case. Sherbrooke [23] proposed METRIC model to study the multi-echelon inventory system. After that, Grave [7] improved the METRIC by assuming the transportation time was a fixed value, and proposed an approximate solving model. In order to solve the problem of lateral supply, Sherbrooke [24] further proposed VARI-METRIC model and developed an

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emulation program. Caglar [3] adopted Heuristic algorithm to minimize system cost subjected to the demand respond constraint.

However, the lack of statistical data brings limitations to these optimization models based on probability theory, which are no longer available. To the best of our knowledge, all those existing methods for inventory forecasting and optimization require massive historical data to build a rational estimation or to satisfy the hypothesis of probability theory. However, it seems unpractical to obtain the statistical information especially for military equipment system due to some constraints, such as technology, time, cost and so on. Therefore, a novel approach to uncertain demand optimization needs to be proposed as barely enough historical data of demand exists.

In 1981, Sommer [25] applied the fuzzy set theory (Zadeh [31]) to inventory optimization with fuzzy dynamic programming method, and thus gradually enriched a new branch of inventory management. However, it was found that once the demand increases from downstream to a high-level depot, the variance of order will be amplified (Ancarani et al. [2]). On one hand, complicacy of inventory management becomes a predominant influence on supply uncertainty; on the other hand, because of the limited cognitions, it is known that human beings always tend to exaggerate or misrepresent the probabilistic information (Kahneman and Tversky [10]).

Under this circumstance, the uncertainty theory was proposed by Liu [16] in 2007 and refined by Liu [15] in 2010 to treat human's belief degree mathematically. Based on normality, duality, subadditivity and product measure axioms, this novel theory has been widely applied in various fields, such as risk analysis (Liu [14]), portfolio selection (Li et al. [20]), facility location-allocation problem (Wen and Iwamura [30]), reliability analysis (Zeng and Wen [32]) and travel itinerary problem (Li et al. [30]). The uncertainty theory is employed in this paper to deal with multi-echelon spare parts supply system.

The rest of this paper will be organized as follows: Firstly, uncertainty theory will be introduced in Section 2 including uncertain variable, uncertainty distribution, and uncertain expected value; Then Section 3 will give a description of the spare part supply system with several rational assumptions and Section 4 will give a list to all notations; The minimal expected backorder model and minimal backorder rate model will be built in Section 5, and Section 6 will employ genetic algorithm to solve these two optimization models; Finally, a numerical example will be given to illustrate the availability and effectiveness of these models.

2. Preliminaries

The concept of uncertain measure was firstly proposed in uncertainty theory by Liu [16] to describe the belief degree of decision makers, which presents the likelihood that an uncertain event happens. Then a concept of uncertain variable was given to model the quantity in uncertain condition, as well as uncertainty distribution, expected value and variance. The interested reader may consult Liu [17].

Let Γ be a nonempty set, and Ł be an σ-algebra over Γ. Define each element $\Lambda \in$ Ł as an event. Liu gave us normality axiom, duality axiom, subadditivity axiom and product axiom in order to assign a number $\mathcal{M}\{\Lambda\} \in [0,1]$ to each event:

Axiom 1. (Liu [16]) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ. (1)

Axiom 2. (Liu [16]) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ. (2)

Axiom 3. (Liu [16]) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{3}$$

Axiom 4. (Liu [16]) Let $(\Gamma_k, Ł_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{4}$$

where Λ_k are arbitrarily chosen events from $Ł_k$ for $k = 1, 2, \cdots$, respectively.

Definition 1. (Liu [16]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, Ł, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\} \tag{5}$$

is an event.

Definition 2. (Liu [16]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \Lambda_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \tag{6}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 3. (Liu [16]) The uncertainty distribution Φ of an uncertain variable ξ is

$$\Phi(x) = \mathcal{M}\{\xi \le x\} \tag{7}$$

for any real number x.

Example 1. An uncertain variable ξ is called discrete if it takes the values in $\{x_1, x_2, \cdots, x_n\}$ and has an uncertain distribution

$$\Phi(x) = \begin{cases} \alpha_0, \text{ if } x < x_1 \\ \alpha_i, \text{ if } x_i \le x < x_{i+1}, 1 \le i < n \\ \alpha_n, \text{ if } x \ge x_n \end{cases} \tag{8}$$

denoted by $\mathcal{D}(x_1, \alpha_1, x_2, \alpha_2, \cdots, x_n, \alpha_n)$ where $x_1 < x_2 < \cdots < x_n$ and $0 = \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n = 1$.

Definition 4. (Liu [16]) An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1.$$

Then its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. In this case, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Example 2. (Liu [16]) An uncertain variable ξ is called zigzag if it takes a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, \text{ if } x \le a \\ (x-a)/2(b-a), \text{ if } a \le x \le b \\ (x+c-2b)/2(c-b), \text{ if } b \le x \le c \\ 1, \text{ if } x \ge c \end{cases} \tag{9}$$

denoted by $\mathcal{Z}(a, b, c)$ where a, b, c are real numbers with $a < b < c$.

Obviously, a zigzag uncertainty distribution is continuous and strictly increasing, and it is regular.

Theorem 1. (Liu [12]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then

$$\xi = f\left(\xi_1, \xi_2, \cdots, \xi_n\right)$$

is an uncertain variable with inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)\right). \tag{10}$$

Example 3. Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Then

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n$$

is an uncertain variable with inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha). \tag{11}$$

Definition 5. (Liu & Ha [18]) The expected value of an uncertain variable ξ is defined by

$$E\left[\xi\right] = \int_0^{+\infty} \mathcal{M}\left\{\xi \geq x\right\} \mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\left\{\xi \leq x\right\} \mathrm{d}x \tag{12}$$

provided that at least one of the two integrals is finite.

Theorem 2. Let ξ be an uncertain variable with an uncertainty distribution Φ. If $E\left[\xi\right]$ exists, then

$$E\left[\xi\right] = \int_0^{+\infty} (1 - \Phi(x))\,\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\,\mathrm{d}x. \tag{13}$$

Example 4. Let $\xi \sim \mathcal{D}(x_1, \alpha_1, x_2, \alpha_2, \cdots, x_n, \alpha_n)$ be a discrete uncertain variable. Then

$$E\left[\xi\right] = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) x_i \tag{14}$$

where $x_1 < x_2 < \cdots < x_n$ and $0 = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n = 1$.

Example 5. Let $\xi \sim Z(a, b, c)$ be a zigzag uncertain variable. Then it has an expected value

$$E\left[\xi\right] = \frac{a + 2b + c}{4}. \tag{15}$$

Theorem 3. (Liu and Ha [18]) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable $\xi = f\left(\xi_1, \xi_2, \cdots, \xi_n\right)$ has an expected value

$$E\left[\xi\right] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)\right) \mathrm{d}\alpha. \tag{16}$$

3. Problem description

3.1. Supply process description

Spare parts inventory is served as a supply system to assist maintenance replacement to keep equipment under operating condition. When failure occurs, the fault component will be replaced with spare part, so that the stock level of spare parts becomes an important factor of significant influence. Therefore, it is a vital target to keep balance between costs for spare parts and readiness capability of equipment system, so as to make the supply adjustable to the lowest level of item cost.

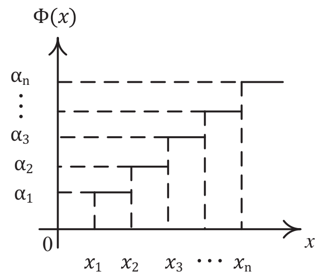

Fig. 1. Discrete Uncertainty Distribution.

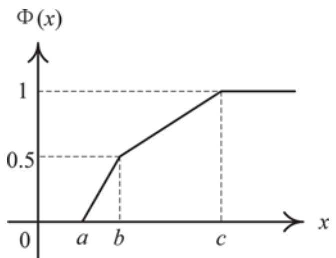

Fig. 2. Zigzag Uncertainty Distribution.

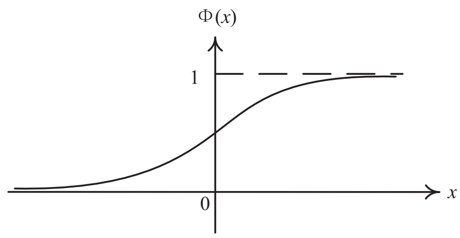

Fig. 3. Zigzag Uncertainty Distribution.

A multi-echelon support system consists of three echelons, namely, organizational level, intermediate level and depot level. Each base at these echelons owns an appropriate inventory depot. Support recourse flows both ways, that is, fault component generated by organizational level is replaced and repaired at this base if possible, or sent to higher level for maintenance when organizational-level maintenance is not available. At the same time, demands for spares generated from lower level are met with spares provided by higher level. The repaired spare part is stored in corresponding inventory depot to the maintenance agency where it is repaired (Figs. 1–4).

Taking a simplified base-depot support system made of one depot and J organizational-level bases for example, it could supplies I items for a complex equipment system. $(s-1, s)$ ordering policy is adopted in this paper. An order request will be sent out from the base when stock level decreases1 item, so as to restore to the original level. Since repairable items hold the characteristics of complex mechanism, high cost and low possibility for fault during the period, the assumption for $(s-1, s)$ ordering policy is appropriate.

Usually support performance is evaluated by efficiency indexes, such as operational availability, expected backorder and so on. Since operational availability can be expressed by the function of expected backorder which is easier to be calculated, expected backorder is chosen as support efficiency index. As another important index, cost spent on support resource mainly involves ordering cost, holding cost and shortage cost caused by out-of-stock downtime.

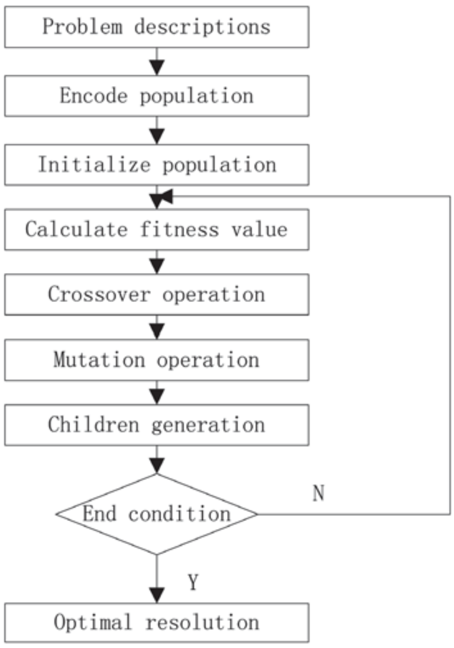

Fig. 4. Genetic Algorithm Procedure.

3.2. Assumption

To concentrate on repairable spare part inventory optimization model itself, several reasonable assumptions are proposed and shown as follows:

1) Demands for each item at each base are independent;
2) Decision on where to repair a fault item depends only on maintenance capacity of bases and maintenance level of the item itself, while stock levels and repair workload are ignored;
3) All items are identically important and as good as new after maintenance;
4) Faults of different units occur without interaction;
5) All units in a stand-alone system are series and any failure can cause a shutdown of the system;
6) Maintenance cost and transit cost are ignored;
7) Maintenance can always be achieved either at base j or at depot;
8) Supplies between bases are forbidden.

3.3. Notations

i: Index of item, $i = 1, 2, \cdots, I$;

j: Index of base, $j = 0, 1, 2, \cdots, J$. It represents a depot when $j = 0$;

x_{ij}: Stock level of item i at base j, $i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J$;

ξ_{ij}: Uncertain demands for item i at base j in a time period $[0, T]$, $i = 1, 2, \cdots, I, j = 1, 2, \cdots, J$;

Φ_{ij}: Uncertainty distribution of ξ_{ij}, $i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J$;

α: Upper limit of supply availability of the overall supply system;

$\bar{C}$: Superior limit of overall cost in spare supply process;

$\bar{B}_0$: Upper limit of backorders of all items at depot;

C:Total cost in spare supply process;

C^o:Total ordering cost;

C^H: Total holding cost;

C^S: Total shortage cost;

c_{ij}^o: Ordering cost per unit of item i at base j,$i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J$;

c_{ij}^H: Holding cost per unit of item i at base j, $i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J$;

c_{ij}^S: Shortage cost per unit of item i at base j, $i = 1, 2, \cdots, I, j = 1, 2, \cdots, J$;

λ_{ij}: Demand rate for spares i at base j, $i = 1, 2, \cdots, I, j = 1, 2, \cdots, J$;

A: Supply availability of all fleets supported by the overall supply system;

A_j: Supply availability of a single flight supported by base j, $j = 1, 2, \cdots, J$;

N_j: Number of equipment at base j, $j = 1, 2, \cdots, J$;

Z_{ij}: Number of units installed in a single equipment, $i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J$;

$\mathscr{M}_j$: Expected backorder rate of item i at base j, $i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J$;

$\bar{r}_j$: Superior limit of backorder rate at base j, $j = 1, 2, \cdots, J$;

$\bar{r}_0$: Superior limit of backorder rate at the depot;

V_i: Volume per unit of item i;

$\bar{v}_j$: Superior limit of volume of all items at base j, $j = 0, 1, 2, \cdots, J$;

W_i: Weight per unit of item i;

$\bar{w}_j$: Superior limit of weight of all items at base j, $j = 0, 1, 2, \cdots, J$.

4. Minimal expected backorder model

The spare parts inventory optimization is to search for the rational stock levels of spare parts with the objective functions such as cost and supply fulfilled rate satisfied. Since sample data about the stock levels of spare parts often absent, no other way can be adopted but to invite the domain experts to obtain their belief degree about uncertain demands. Therefore, the belief degree about the demands for spare parts at each base is regarded as an uncertainty distribution in the framework of uncertainty theory, and formulates a minimal expected backorder model.

4.1. Objective function

As analyzed above, expected backorder is regarded as our objective function. A spare part backorder will be sent out when the quantity of demands exceeds the stock levels of spares. The shortage can be supplied with a spare from warehouse or an ordered one from higher depot. Therefore, expected backorder is indicated with different expected value between demands and stock level. What should be noted is that backorder at depot is ignored during the calculation, because it refers to backorder at every base that influences the availability of equipment system rather than the one at depot. The objective function can be expressed as

$$F(\boldsymbol{x}) = \min \sum_{j=1}^{J} \sum_{i=1}^{I} \left[\left(E\left[\xi_{ij}\right] - x_{ij} \right) \vee 0 \right] \tag{17}$$

where ξ_{ij} is uncertain variable and x_{ij} is decision variable.

4.2. Constraint 1

Cost C is regarded as a major constraint on optimization objective including ordering cost C^o holding cost C^H and shortage cost C^S.

It could be easily concluded that $C = C^o + C^H + C^S$. The ordering cost can be easily expressed as

$$C^o = \sum_{j=0}^{J} \sum_{i=1}^{I} c_{ij}^o x_{ij}. \tag{18}$$

Due to gradually declining stock level, the holding cost can be derived by line integral and presented as

$$C^H = \sum_{j=0}^{J}\sum_{i=1}^{I}\left(x_{ij}c_{ij}^H T - 0.5\lambda_{ij}c_{ij}^H T^2\right) \quad ,(19)$$

where λ_{ij} is defined as demand rate for spares i at base j, namely the average demand in every year. Thus there exists the fact that $\lambda_{ij} = \xi_{ij}/T$, in which $j = 1, 2, \cdots, J$. Notice that demand rate for item i at depot λ_{i0} is a function of the backorders generated by bases. The above expression can be transformed into

$$C^H = \sum_{j=1}^{J}\sum_{i=1}^{I}\left(x_{ij}c_{ij}^H T - 0.5\xi_{ij}c_{ij}^H T\right) + \sum_{i=1}^{I}\left(x_{i0}c_{i0}^H T - 0.5\left\{\sum_{j=1}^{J}\left[\left(\xi_{ij} - x_{ij}\right)\vee 0\right]\right\}c_{i0}^H T\right). \quad (20)$$

Since the holding cost C^H includes uncertain variables which cannot be compared directly, we can use their expected values instead:

$$E\left[C^H\right] = \sum_{j=1}^{J}\sum_{i=1}^{I}\left(x_{ij}c_{ij}^H T - 0.5E\left[\xi_{ij}\right]c_{ij}^H T\right) + \sum_{i=1}^{I}\left(x_{i0}c_{i0}^H T - 0.5\left\{\sum_{j=1}^{J}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right]\right\}c_{i0}^H T\right).$$

Since shortage cost caused by out-of-stock downtime can bring extensive damages especially during wartime, it is assumed that it grows at an exponential rate with expected backorders. Thus the following formula can be easily obtained:

$$C^S = \sum_{j=1}^{J}\sum_{i=1}^{I}c_{ij}^{\hat{S}}\left(E\left[\xi_{ij}\right] - x_{ij}\right) \quad (21)$$

In the formula, $c_{ij}^o, c_{ij}^H, c_{ij}^S, T$ are already known.

4.3. Constraint 2

Another index employed as constraint is supply availability. To reflect the support performance of supply system, the supply availability is adopted in this paper, which directly reflects the support performance in the supply chains. It indicates the expected value of the percentage of equipment which cease to work caused not by spare shortage. The supply availability of a single flight supported by base can be characterized as

$$A_j = \prod_{i=1}^{I}\left[1 - \left(E\left[\xi_{ij}\right] - x_{ij}\right)/\left(N_j Z_{ij}\right)\right]^{Z_{ij}}. \quad (22)$$

4.4. Constraint 3

As is shown before, backorders at depot are ignored while calculating the backorders of the supply performance of the overall spare support system. However, the backorders at depot influence the supply system a lot; it must be taken into consideration in uncertain optimization model, so we take the backorders of depot as another constraint, which can be expressed as

$$\sum_{i=1}^{I}\left\{\sum_{j=1}^{J}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right] - x_{i0}\right\} \le \bar{B}_o. \quad (23)$$

4.5. Constraint 4

The last inequality is added for weight, volume and other external constraints from external environment. Note that stock levels for all items are positive integers.

Therefore, minimal expected backorder model can be expressed as follows:

$$\begin{cases} \min\sum_{j=1}^{J}\sum_{i=1}^{I}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right] \\ s.t. \\ \sum_{j=0}^{J}\sum_{i=1}^{I}\left(c_{ij}^o + c_{ij}^H T\right)x_{ij} + \sum_{j=1}^{J}\sum_{i=1}^{I}\left\{c_{ij}^{S(E[\xi_{ij}]-x_{ij})} \right. \\ \quad \left. -0.5T\left\{c_{ij}^H E\left[\xi_{ij}\right] + c_{i0}^H\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right]\right\}\right\} \le \bar{C} \\ \sum_{j=1}^{J}\left\{\prod_{i=1}^{I}\left[1 - \left(E\left[\xi_{ij}\right] - x_{ij}\right)/\left(N_j Z_{ij}\right)\right]^{Z_{ij}}\right\}N_j/\left(\sum_{j=1}^{J}N_j\right) \ge \alpha \\ \sum_{i=1}^{I}\left\{\sum_{j=1}^{J}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right] - x_{i0}\right\} \le \bar{B}_o \\ x_{ij} \in N, i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J \\ g_k(x) \le 0, k = 1, 2, \cdots, p. \end{cases} \quad (24)$$

5. Minimal backorder rate model

Backorder rate is another efficient index to evaluate the performance of support system. Charnes and Cooper [4] proposed the chance-constrained programming, which can be utilized to handle the problem with a given confidence level subjected to the chance constraints. The second model based on the chance-constrained programming model is established to minimize the backorder rate.

5.1. Objective function

In this model, uncertainty measure is employed to indicate the possibilities that the shortage of spare parts cannot be supplied with stock level x_{ij} of item i as soon as the fault occurs. Since performance of the whole support system is more important than one of a single base, backorder rate is calculated with considering uncertain measures that backorder of any item generates at all bases. Thus the objective function can be expressed as

$$G(\boldsymbol{x}) = \min_{\boldsymbol{x}}\sum_{j=1}^{J}\sum_{i=1}^{I}\mathscr{M}\left\{\xi_{ij} \ge x_{ij}\right\} \quad (25)$$

In the formula, $\mathscr{M}\left\{\xi_{ij} \ge x_{ij}\right\}$ is an uncertain measure.

5.2. Constraint functions

Despite the fact that only backorders generated from bases directly affect the performance of equipment system, it is necessary to restrain the backorder rate of items at one base or depot. When considering the backorder rate at base j, formula can be obtained as follows:

$$\sum_{i=1}^{I} \mathcal{M}\left\{\xi_{ij} \geq x_{ij}\right\} \leq \bar{r}_j, j = 1, 2, \cdots, J. \quad (26)$$

It can be easily found that the cost constraint and the supply availability constraint are similar with those presented in the first model; other factors needed to be considered are listed with the last inequality constraints.

Above all, the following minimal backorder rate model can be obtained:

$$\begin{cases} \min\limits_{x} \sum_{j=1}^{J}\sum_{i=1}^{I} \mathcal{M}\left\{\xi_{ij} \geq x_{ij}\right\} \\ s.t. \\ \sum_{j=0}^{J}\sum_{i=1}^{I}\left(c_{ij}^{o} + c_{ij}^{H}T\right)x_{ij} + \sum_{j=1}^{J}\sum_{i=1}^{I}\left\{c_{ij}^{S(E[\xi_{ij}]-x_{ij})}\right. \\ \qquad \left. -0.5T\left\{c_{ij}^{H}E\left[\xi_{ij}\right] + c_{i0}^{H}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right]\right\}\right\} \leq \bar{C} \\ \sum_{j=1}^{J}\left\{\prod_{i=1}^{I}\left[1 - \left(E\left[\xi_{ij}\right] - x_{ij}\right)/\left(N_j Z_{ij}\right)\right]^{Z_{ij}}\right\} N_j/\left(\sum_{j=1}^{J} N_j\right) \geq \alpha \\ \sum_{i=1}^{I} \mathcal{M}\left\{\xi_{ij} \geq x_{ij}\right\} \leq \bar{r}_j, j = 1, 2, \cdots, J \\ \sum_{i=1}^{I}\left\{\sum_{j=1}^{J}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right] - x_{i0}\right\} \leq \bar{B}_o \\ x_{ij} \in N, i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J \\ g_k(x) \leq 0, k = 1, 2, \cdots, p. \end{cases} \quad (27)$$

It can be easily proved that the model is equivalent to the following one:

$$\begin{cases} \min\limits_{x} \sum_{j=1}^{J}\sum_{i=1}^{I} \left(1 - \Phi\left(x_{ij}\right)\right) \\ s.t. \\ \sum_{j=0}^{J}\sum_{i=1}^{I}\left(c_{ij}^{o} + c_{ij}^{H}T\right)x_{ij} + \sum_{j=1}^{J}\sum_{i=1}^{I}\left\{c_{ij}^{S(E[\xi_{ij}]-x_{ij})}\right. \\ \qquad \left. -0.5T\left\{c_{ij}^{H}E\left[\xi_{ij}\right] + c_{i0}^{H}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right]\right\}\right\} \leq \bar{C} \\ \sum_{j=1}^{J}\left\{\prod_{i=1}^{I}\left[1 - \left(E\left[\xi_{ij}\right] - x_{ij}\right)/\left(N_j Z_{ij}\right)\right]^{Z_{ij}}\right\} N_j/\left(\sum_{j=1}^{J} N_j\right) \geq \alpha \\ \sum_{i=1}^{I} \left(1 - \Phi\left(x_{ij}\right)\right) \leq \bar{r}_j, j = 1, 2, \cdots, J \\ \sum_{i=1}^{I}\left\{\sum_{j=1}^{J}\left[\left(E\left[\xi_{ij}\right] - x_{ij}\right)\vee 0\right] - x_{i0}\right\} \leq \bar{B}_o \\ x_{ij} \in N, i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J \\ g_k(\boldsymbol{x}) \leq 0, k = 1, 2, \cdots, p. \end{cases} \quad (28)$$

6. A Special case

As we discussed in Section 2, if $c_1, c_2, \cdots, c_Q$ are nonnegative numbers and $c_1 + c_2 + \cdots + c_Q = 1$, the expected uncertain variable can be calculated with the formula as follows:

$$E\left[\xi_{ij}\right] = \sum_{q=1}^{Q} c_{qij}\xi_{qij}. \quad (29)$$

Expected value operator of uncertain variable and uncertain expected value model was proposed by Liu [16] in 2007. Li [19] has given some researches to expected values of strictly monotone functions, and we will also give its equivalent model which can be obtained as follows:

$$\begin{cases} \min \sum_{j=1}^{J}\sum_{i=1}^{I}\left[\left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)\vee 0\right] \\ s.t. \\ \sum_{j=0}^{J}\sum_{i=1}^{I}\left(c_{ij}^{o} + c_{ij}^{H}T\right)x_{ij} \\ +\sum_{j=1}^{J}\sum_{i=1}^{I}\left\{c_{ij}^{S}\left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)\right. \\ \qquad \left. -0.5T\left\{c_{ij}^{H}\sum_{q=1}^{Q} c_{qij}\xi_{qij} + c_{i0}^{H}\left[\left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)\vee 0\right]\right\}\right\} \leq \bar{C} \\ \sum_{j=1}^{J}\left\{\prod_{i=1}^{I}\left[1 - \left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)/\left(N_j Z_{ij}\right)\right]^{Z_{ij}}\right\} N_j/\left(\sum_{j=1}^{J} N_j\right) \geq \alpha \\ \sum_{i=1}^{I}\left\{\sum_{j=1}^{J}\left[\left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)\vee 0\right] - x_{i0}\right\} \leq \bar{B}_o \\ x_{ij} \in N, i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J \\ g_k(x) \leq 0, k = 1, 2, \cdots, p. \end{cases} \quad (30)$$

The conclusion from Section 2 can be got if $c_1, c_2, \cdots, c_Q$ are nonnegative numbers, and if $c_1 + c_2 + \cdots + c_Q = 1$, then

$$\Phi\left(x_{ij}\right) = \sum_{\xi_{qij} \leq x_{ij}} c_{qij} \quad . (31)$$

Based on chance constraint programming model proposed by Liu [13], the following formula can be obtained:

$$\begin{cases} \min\limits_{x} \sum_{j=1}^{J}\sum_{i=1}^{I}\left(1 - \sum_{\xi_{qij} \leq x_{ij}} c_{qij}\right) \\ s.t. \\ \sum_{j=0}^{J}\sum_{i=1}^{I}\left(c_{ij}^{o} + c_{ij}^{H}T\right)x_{ij} \\ +\sum_{j=1}^{J}\sum_{i=1}^{I}\left\{c_{ij}^{S}\left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)\right. \\ \qquad \left. -0.5T\left\{c_{ij}^{H}\sum_{q=1}^{Q} c_{qij}\xi_{qij} + c_{i0}^{H}\left[\left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)\vee 0\right]\right\}\right\} \leq \bar{C} \\ \sum_{j=1}^{J}\left\{\prod_{i=1}^{I}\left[1 - \left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)/\left(N_j Z_{ij}\right)\right]^{Z_{ij}}\right\} N_j/\left(\sum_{j=1}^{J} N_j\right) \geq \alpha \\ \sum_{i=1}^{I}\left(1 - \sum_{\xi_{qij} \leq x_{ij}} c_{qij}\right) \leq \bar{r}_j, j = 1, 2, \cdots, J \\ \sum_{i=1}^{I}\left\{\sum_{j=1}^{J}\left[\left(\sum_{q=1}^{Q} c_{qij}\xi_{qij} - x_{ij}\right)\vee 0\right] - x_{i0}\right\} \leq \bar{B}_o \\ x_{ij} \in N, i = 1, 2, \cdots, I, j = 0, 1, 2, \cdots, J \\ g_k(\boldsymbol{x}) \leq 0, k = 1, 2, \cdots, p. \end{cases} \quad (32)$$

7. Genetic algorithm

In the previous researches, spare parts inventory optimization models were solved through integer programming, dynamic programming and many other heuristic algorithms. However, the minimal expected backorder model and minimal backorder rate model are nonlinear discrete programming models. The genetic algorithm is a search heuristic that efficiently solves nonlinear problems and discrete problems. The advantage of this algorithm lies in global optimization and it can flexibly adjust its search direction without determined rules, so that the genetic algorithm is employed to solve these two models. The procedure of searching optimal stock levels of spare parts is implemented by the representation of initial population, fitness evaluation, genetic operation (selection, crossover and mutation). This process can be described as follows.

7.1. Problem representation

A chromosome V_p is defined as an integer vector of stock levels of spare parts for all items at bases and depot to represent a solution $\boldsymbol{x}$. Thus the $p-th$ chromosome can be expressed as

$V_p=\left(x_{p10},x_{p11},\cdots,x_{p1J},\cdots,x_{pI0},x_{pI1},\cdots,x_{pIJ}\right)$.

The chromosome of initial population can be generated through a random seed approach, N for the upper limit of stock levels of spare parts, until *pop_size* chromosomes are generated, $p=1,2,\cdots,pop\_size$.

7.2. Fitness calculation

Calculate the objective value B^p for each chromosome V_p, and then the fitness value *eval*. The rank-based evaluation function is defined as

$Eval\left(V_p\right)=\alpha(1-\alpha)^{p-1}, p=1,2,\cdots,pop\_size,$

where $V_1,V_2,\cdots,V_{pop\_size}$ are assumed to have been rearranged from good to bad according to their objective values B^p and $\alpha\in(0,1)$ is a parameter in the genetic system.

7.3. Crossover operation

Rearrange chromosomes of each population according to their fitness from big to small, and the first chromosome pass on directly to the next generation. Other chromosomes are selected from generation based on spinning roulette wheel characterized by fitness for *pop_size* times. Each time we select a single chromosome.

Then define a parameter P_c as the probability of crossover. Generate a random number θ from the interval $[0,1]$, and the chromosome V_p is selected if $\theta<P_c$. Define selected parents as $\boldsymbol{x}'_1,\boldsymbol{x}'_2,\cdots$. The children of $\boldsymbol{x}'_1$ and $\boldsymbol{x}'_2$ are

$$\boldsymbol{x}_1''=\left(x_{110}^{(1)}\times w+x_{110}^{(2)}\times(1-w),x_{111}^{(1)}\times w+x_{111}^{(2)}\times(1-w),\cdots,x_{1IJ}^{(1)}\times w+x_{1IJ}^{(2)}\times(1-w)\right)$$

and

$$\boldsymbol{x}_2''=\left(x_{210}^{(2)}\times w+x_{210}^{(1)}\times(1-w),x_{211}^{(2)}\times w+x_{211}^{(1)}\times(1-w),\cdots,x_{2IJ}^{(2)}\times w+x_{2IJ}^{(1)}\times(1-w)\right).$$

Check the feasibility for each child before accepting it. If both children are feasible, then we replace the parents with the children. If not, we keep the existing feasible ones, and then redo the crossover operator until two feasible children are obtained or a number of cycles are finished.

Table 1
Parameters of All Items at All Bases and Depot.

	V_i	W_i	Unit Cost c_i($M/unit)			N_j	Z_{ij}
			c_{ij}^o	c_{ij}^H	c_{ij}^S		
Base1 Item1	1	1	1	0.2	$\hat{3}\left(E\left[\xi_{11}\right]-x_{11}\right)$	12	6
Base1 Item2	0.8	0.6	1.5	0.3	$\hat{3}\left(E\left[\xi_{21}\right]-x_{21}\right)$		4
Base2 Item1	1	1	1	0.2	$\hat{3}\left(E\left[\xi_{12}\right]-x_{12}\right)$	10	6
Base2 Item2	0.8	0.6	1.5	0.3	$\hat{3}\left(E\left[\xi_{22}\right]-x_{22}\right)$		4
Depot Item1	1	1	1	0.15			
Depot Item2	0.8	0.6	1.5	0.2			

Table 2
Parameters Constraints of All Items at All Bases and Depot.

	T	α	$\bar{C}$	$\bar{B}_o$	$\bar{r}_j$	$\bar{v}_j$	$\bar{w}_j$
Base1	0.1		700		0.25	1	1
Base2					0.25	0.8	0.6
Depot				5		1	1

7.4. Mutation operation

Define a parameter P_m as the probability of mutation. Generate a random number ρ from the interval $[0,1]$, and the chromosome V_p is selected if $\rho<P_m$. Define U as an appropriate large positive number. The child of x_1 is

$$\boldsymbol{x}_1'=\left(x_{110}+d_{110}\times u,x_{111}+d_{111}\times u,\cdots,x_{1IJ}+d_{1IJ}\times u\right),$$

in which $d_{kij}\in[-1,1]$, $u\in[0,U]$. If $\boldsymbol{x}'$ is not feasible, then set W as a random number between 0 and U until it is feasible.

7.5. Algorithm description

Step 1. From the potential region initialize *pop_size* chromosomes $V_p=\left(x_{p10},x_{p11},\cdots,x_{p1J},\cdots,x_{pI0},x_{pI1},\cdots,x_{pIJ}\right)$, which denote stock levels of spare parts for all items at bases and depot.

Step 2. Calculate the objective values U for all chromosomes B^p, $p=1,2,\cdots,pop\_size$.

Step 3. Compute the fitness of all chromosomes B^p, $p=1,2,\cdots,pop\_size$.

Step 4. Select the chromosomes for a new population.

Step 5. Renew the chromosomes B^p, $p=1,2,\cdots,pop\_size$ through crossover operation.

Step 6. Update the chromosomes B^p, $p=1,2,\cdots,pop\_size$ through mutation operation.

Step 7. Repeat the second to the sixth steps for a given number of cycles.

Step 8. Report the best chromosome $V^*=\boldsymbol{x}^*$ as the optimal stock levels of all items supplied by support system.

8. Numerical example

Consider a spare part supply system consisting of 1 depot and 2 bases: each base supplies 2 items A and B for an air fleet, while the depot supplies items A and B for these two bases. Uncertain demand distribution functions for the two parts are figured out according to the belief degree from ten experts. In this paper, the constraints of volume and weight of the spare parts at each base and depot will be fully taken into consideration. The costs consist of three parts including ordering cost, holding cost and shortage cost. Initial support time period is 0.1 year. Parameters of items at all bases and depot in details are given in Tables 1 and 2.

Then uncertain spare part optimization models introduced in Section 4 and Section 5 can be adopted to give the optimal stock level allocation with x_{ij} for item i at base j. Supposing that all the demands are discrete uncertainty variables introduced in Example 1, initial uncertain distributions of demands for each item at each base are given as follows:

$$\Phi\left(\xi_{11}\right)=\begin{cases}0, if \xi_{11}<62\\ 0.1, if 62\le\xi_{11}<66\\ 0.2, if 66\le\xi_{11}<68\\ 0.5, if 68\le\xi_{11}<72\\ 0.7, if 72\le\xi_{11}<77\\ 0.9, if 77\le\xi_{11},\end{cases}$$

$$\Phi\left(\xi_{12}\right)=\begin{cases}0, if \xi_{12}<60\\ 0.1, if 60\le\xi_{12}<63\\ 0.2, if 63\le\xi_{12}<66\\ 0.5, if 66\le\xi_{12}<71\\ 0.7, if 71\le\xi_{12}<76\\ 0.9, if 76\le\xi_{12},\end{cases}$$

$$\Phi\left(\xi_{21}\right)=\begin{cases}0, if \xi_{21}<40\\ 0.1, if 40\le\xi_{21}<42\\ 0.2, if 42\le\xi_{21}<45\\ 0.5, if 45\le\xi_{21}<47\\ 0.7, if 47\le\xi_{21}<54\\ 0.9, if 54\le\xi_{21},\end{cases}$$

$$\Phi\left(\xi_{22}\right)=\begin{cases}0, if \xi_{22}<38\\ 0.1, if 38\le\xi_{22}<40\\ 0.2, if 40\le\xi_{22}<43\\ 0.5, if 43\le\xi_{22}<46\\ 0.7, if 46\le\xi_{22}<53\\ 0.9, if 53\le\xi_{22}.\end{cases}$$

After that, the minimal expected backorder model can be obtained as follows:

$$\begin{cases}\min\sum_{j=1}^{J}\sum_{i=1}^{I}\left[\left(\sum_{q=1}^{Q}c_{qij}\xi_{qij}-x_{ij}\right)\vee 0\right]\\ s.t.\\ \sum_{j=0}^{J}\sum_{i=1}^{I}\left(c_{ij}^{o}+c_{ij}^{H}T\right)x_{ij}\\ +\sum_{j=1}^{J}\sum_{i=1}^{I}\left\{c_{ij}^{\hat{S}}\left(\sum_{q=1}^{Q}c_{qij}\xi_{qij}-x_{ij}\right)\right.\\ \qquad\left.-0.5T\left\{c_{ij}^{H}\sum_{q=1}^{Q}c_{qij}\xi_{qij}+c_{i0}^{H}\left[\left(\sum_{q=1}^{Q}c_{qij}\xi_{qij}-x_{ij}\right)\vee 0\right]\right\}\right\}\le\bar{C}\\ \sum_{j=1}^{J}\left\{\prod_{i=1}^{I}\left[1-\left(\sum_{q=1}^{Q}c_{qij}\xi_{qij}-x_{ij}\right)/\left(N_jZ_{ij}\right)\right]^{Z_{ij}}\right\}N_j/\left(\sum_{j=1}^{J}N_j\right)\ge\alpha\\ \sum_{i=1}^{I}\left\{\sum_{j=1}^{J}\left[\left(\sum_{q=1}^{Q}c_{qij}\xi_{qij}-x_{ij}\right)\vee 0\right]-x_{i0}\right\}\le\bar{B}_o\\ \sum_{i=1}^{I}V_ix_{ij}\le\bar{v}_j, j=0,1,2,\cdots,J\\ \sum_{i=1}^{I}W_ix_{ij}\le\bar{w}_j, j=0,1,2,\cdots,J\\ x_{ij}\in N, i=1,2,\cdots,I, j=0,1,2,\cdots,J.\end{cases}\tag{37}$$

In order to solve this model, the genetic algorithm has run with 10,000 generations. Optimal solution of the expected model is shown in Table 3.

Table 3
Optimal stock levels based on MEBO model.

Stock Levels	Depot (Base 0)	Base 1	Base 2
Item 1	3	57	58
Item 2	2	44	42

Table 4
Optimal stock levels based on MBRO model.

Stock Levels	Depot (Base 0)	Base 1	Base 2
Item 1	2	58	58
Item 2	2	44	42

Similarly, the minimal backorder rate model can be obtained, which is omitted in this example. The optimal solution is shown in Table 4.

It can be found from Table 4 that the results of two models are generally agreeable. There exists a minute difference between two results, which is caused by the fact that the data of constraints given as known parameters in the two models are not entirely consistent, and it exerts no influence on the availability of the two models. Above all, the models are effective in solving optimization problems with statistic data.

9. Conclusion

This paper mainly presented recent studies about spare part inventory optimization problem with uncertain demands; a new method was introduced to deal with this issue. A minimal expected backorder model and a minimal backorder rate model have been proposed in this paper, which are based on a multi-echelon inventory supply system. Meanwhile, some equivalent crisp models and optimal solutions with genetic algorithm have been obtained. Obviously, it becomes achievable to optimize the stock level of spares without massive statistic data. And it has been proved that the uncertain optimization models are viable and effective through the last numerical example.

Our main purpose is to give a novel sight of modeling in uncertainty conditions. This paper just provided a starting point in spare parts supply system with application of uncertainty theory. Some conditions were inevitably simplified, such as the no lateral supply assumption, to focus on the essence of the problem with ignoring side conditions. Therefore, more endeavors are still needed to improve the models to adapt to the complex application environment.

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Reliability analysis in uncertain random system

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Abstract Reliability analysis of a system based on probability theory has been widely studied and used. Nevertheless, it sometimes meets with one problem that the components of a system may have only few or even no samples, so that we cannot estimate their probability distributions via statistics. Then reliability analysis of a system based on uncertainty theory has been proposed. However, in a general system, some components of the system may have enough samples while some others may have no samples, so the reliability of the system cannot be analyzed simply based on probability theory or uncertainty theory. In order to deal with this type systems, this paper proposes a method of reliability analysis based on chance theory which is a generalization of both probability theory and uncertainty theory. In order to illustrate the method, some common systems are considered such as series system, parallel system, k-out-of-n system and bridge system.

Keywords Uncertainty theory · Uncertain random variable · Chance measure · Reliability · Boolean system

1 Introduction

System reliability analysis plays a crucial role in engineering since the occurrence of failures maybe lead to catastrophic consequences. Most researchers assumed each

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component in the system works with a given probability and studied the system reliability from mathematical aspect. In 1947, Freudethal first developed structural reliability that is the application of probabilistic methods. Then Cornell (1969) proposed a structural reliability index. From then on, reliability analysis based on probability theory has got many significant achievements.

Before applying probability theory to practical problem, a fundamental premise is to estimate probability distribution that is close enough to frequency. Otherwise, the law of large numbers is no longer valid. In fact, we sometimes have no observed date because of the technological or economical difficulties. In this case, we have to invite the experts to evaluate their belief degree that a component works well. However, Liu (2015) pointed that *human beings usually estimate a much wider range of values than the object actually takes*. Therefore, the belief degrees deviate far from the frequency. If we still take human belief degrees as probability distribution, we maybe cause a counterintuitive result that was given by Liu (2012).

In order to model the belief degree, an uncertainty theory was founded by Liu (2007). It satisfies normality, duality, subadditivity and product axioms in mathematics. Nowadays, uncertainty theory has become a branch of pure mathematics and has been widely applied in many fields such as uncertain programming (Liu 2009a, b), uncertain risk analysis (Liu 2010a, b), and uncertain reliability analysis (Liu 2010b).

In a general system, some components may have enough samples to ascertain their functioning probabilities, while some others may have no samples. In order to deal with this phenomenon, Liu (2013a) proposed chance theory as a mixture of probability theory and uncertainty theory in 2013. After that, chance theory was developed steadily and applied widely in many fields such as uncertain random programming (Liu 2013b; Ke et al. 2014; Zhou et al. 2014), uncertain risk analysis (Liu and Ralescu 2014, 2016), uncertain random graph (Liu 2014), and uncertain random network (Liu 2014; Sheng and Gao 2014).

Probability theory is applicable when we have a large amount of samples, and uncertainty theory is applicable when we have no samples but belief degree from the experts. Chance theory, as a mixture of probability theory and uncertainty theory, is applicable for a complex system containing uncertainty and randomness. In this paper, we aim at employing chance theory to analyze the reliability of a complex system involving both uncertainty and randomness. The rest of this paper is organized as follows. Section 2 introduces some basic concepts about uncertain variable and uncertain random variable. Section 3 proposes the concept of reliability index of an uncertain random system, and the reliability of a series system and a parallel system will be analyzed. Section 4 proves a reliability index theorem. A k-out-of-n system, parallel–series system, series–parallel system and bridge system are studied. At last, some conclusions are made in Sect. 5.

2 Preliminaries

In this section, we introduce some basic concepts and results in uncertainty theory and chance theory.

2.1 Uncertainty theory

Uncertainty theory was founded by Liu (2007) and refined by Liu (2010a). Mathematically, uncertainty theory satisfies normality, duality, subadditivity and product axioms. Practically, uncertainty is anything that is described by belief degrees.

Definition 2.1 (Liu 2007) Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following three axioms,

Axiom 1 $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2 $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$.
Axiom 3 For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$ we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

In this case, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Besides, in order to provide the operational law, another axiom named product axiom was proposed by Liu (2009a).

Axiom 4 Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.2 (Liu 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, we have

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \in \mathcal{L}.$$

Definition 2.3 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

If the uncertainty distribution $\Phi(x)$ of ξ has an inverse function $\Phi^{-1}(\alpha)$ for $\alpha \in (0, 1)$, then ξ is called a regular uncertain variable, and $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ. Inverse uncertainty distribution plays an important role in the operations of independent uncertain variables.

Definition 2.4 (Liu 2007) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 2.1 (Liu 2010a) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Assume the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$, and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$. Then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right).$$

An uncertain variable is called Boolean if it takes values either 0 or 1. The following is a Boolean uncertain variable

$$\xi = \begin{cases} 1 & \text{with uncertain measure } a \\ 0 & \text{with uncertain measure } 1-a \end{cases}$$

where $a \in [0, 1]$. The operational law of Boolean system was introduced by Liu (2010a) as follows.

Theorem 2.2 (Liu 2010a) *Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} 1 & \textit{with uncertain measure } a_i \\ 0 & \textit{with uncertain measure } 1-a_i \end{cases}$$

for $i = 1, 2, \ldots, n$. If f is a Boolean function, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), & \textit{if} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\ldots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), & \textit{if} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \textit{if } x_i = 1 \\ 1-a_i, & \textit{if } x_i = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$, respectively.

Definition 2.5 (Liu 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}\mathrm{d}r.$$

For an uncertain variable ξ with uncertainty distribution $\Phi(x)$, its expected value can be expressed as

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x.$$

And if ξ has an inverse uncertainty distribution function $\Phi^{-1}(\alpha)$, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Theorem 2.3 (Liu and Ha 2010) *Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right)\mathrm{d}\alpha$$

provided that $E[\xi]$ exists.

2.2 Chance theory

Chance theory, as a mixture of uncertainty theory and probability theory, was founded by Liu (2013a, b) to deal with a system exhibiting both randomness and uncertainty. The basic concept is the chance measure of an uncertain random event in a chance space.

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr) = (\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \Pr)$$

is called a chance space.

Definition 2.6 (Liu 2013a) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure Ch of Θ is defined by

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq r\}\mathrm{d}r.$$

Theorem 2.4 (Liu 2013a) *Let* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ *be a chance space. Then the chance measure* $\mathrm{Ch}\{\Theta\}$ *is a monotone increasing function of* Θ *and*

$$\mathrm{Ch}\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times \Pr\{A\}$$

for any $\Lambda \in \mathcal{L}$ *and any* $A \in \mathcal{A}$. *Especially, we have*

$$\mathrm{Ch}\{\emptyset\} = 0, \quad \mathrm{Ch}\{\Gamma \times \Omega\} = 1.$$

Definition 2.7 (Liu 2013a) An uncertain random variable ξ is a measurable function from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e.,

$$\{\xi \in B\} = \{(\gamma, \omega) \mid \xi(\gamma, \omega) \in B\}$$

is an uncertain random event for any Borel set B.

When an uncertain random variable $\xi(\gamma, \omega)$ does not vary with γ, it degenerates to a random variable. When an uncertain random variable $\xi(\gamma, \omega)$ does not vary with ω, it degenerates to an uncertain variable. Therefore, a random variable and an uncertain variable are two special uncertain random variables.

Example 2.1 Let $\xi_1, \xi_2, \ldots, \xi_m$ be random variables and $\eta_1, \eta_2, \ldots, \eta_n$ be uncertain variables. If f is a measurable function, then

$$\tau = f(\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_n)$$

is an uncertain random variable determined by

$$\tau(\gamma, \omega) = f(\xi_1(\omega), \xi_2(\omega), \ldots, \xi_m(\omega), \eta_1(\gamma), \eta_2(\gamma), \ldots, \eta_n(\gamma))$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Definition 2.8 (Liu 2013a) Let ξ be an uncertain random variable. Then its chance distribution is defined by

$$\Phi(x) = \mathrm{Ch}\{\xi \le x\}$$

for any $x \in \Re$.

As two special uncertain random variables, the chance distribution of a random variable ξ is just its probability distribution

$$\Phi(x) = \mathrm{Ch}\{\xi \le x\} = \Pr\{\xi \le x\},$$

and the chance distribution of an uncertain variable ξ is just its uncertainty distribution

$$\Phi(x) = \mathrm{Ch}\{\xi \le x\} = \mathcal{M}\{\xi \le x\}.$$

Theorem 2.5 (Liu 2013b) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, respectively, and $\tau_1, \tau_2, \ldots, \tau_n$ be uncertain variables. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \ldots, y_m)\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\ldots\mathrm{d}\Psi_m(y_m)$$

where $F(x; y_1, y_2, \ldots, y_m)$ is the uncertainty distribution of the uncertain variable

$$f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)$$

for any real numbers $y_1, \ldots, y_m$.

Definition 2.9 (Liu 2013a) Let ξ be an uncertain random variable. Then its expected value is

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Ch}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

For an uncertain random variable ξ with chance distribution $\Phi(x)$, its expected value can be briefed as

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x.$$

If $\Phi(x)$ is regular, then

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Theorem 2.6 (Liu 2013b) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \ldots, \tau_n$ be uncertain variables. Then the uncertain random variable*

$$\xi = f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$$

has an expected value

$$E[\xi] = \int_{\Re^m} E[f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)]\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_m(y_m)$$

where $E[f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)]$ is the expected value of the uncertain variable $f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)$ for any given real numbers $y_1, \ldots, y_m$.

3 Reliability of uncertain random system

A function f is called a Boolean function if it maps $\{0, 1\}^n$ to $\{0, 1\}$. It is usually used to model the structure of a Boolean system.

Definition 3.1 Assume that a Boolean system ξ is comprised of n components $\xi_1, \xi_2, \ldots, \xi_n$. Then a Boolean function f is called its structure function if

$$\xi = 1 \quad \text{if and only if } f(\xi_1, \xi_2, \ldots, \xi_n) = 1. \tag{1}$$

Obviously, when f is the structure function of the system, we also have $\xi = 0$ if and only if $f(\xi_1, \xi_2, \ldots, \xi_n) = 0$. For a series system containing n components, the structure function is

$$f(\xi_1, \ldots, \xi_n) = \bigwedge_{i=1}^{n} \xi_i.$$

For a parallel system containing n components, the structure function is

$$f(\xi_1, \ldots, \xi_n) = \bigvee_{i=1}^{n} \xi_i.$$

For a k-out-of-n system, the structure function is

$$f(\xi_1, \ldots, \xi_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} \xi_i \geq k \\ 0, & \text{if } \sum_{i=1}^{n} \xi_i < k. \end{cases}$$

In a complex system, some components may have enough samples to estimate their probability distributions, and can be regarded as random variables, while some others may have no samples, and can only be evaluated by the experts and regarded as uncertain variables. In this case, the system cannot be simply modeled by a stochastic system or an uncertain system. Then we will employ uncertain random variable to model the system, and analyze its reliability based on chance theory.

Definition 3.2 The reliability index of an uncertain random system ξ is defined as the chance measure that the system is working, i.e.,

$$Reliability = \text{Ch}\{\xi = 1\}. \tag{2}$$

If all uncertain random components degenerate to random ones, then the reliability index is the probability measure that the system is working. If all uncertain random components degenerate to uncertain ones, then the reliability index (Liu 2010b) is the uncertain measure that the system is working.

Example 3.1 (Series System) Consider a series system containing independent random components $\xi_1, \xi_2, \ldots, \xi_m$ with reliabilities $a_1, a_2, \ldots, a_m$, and independent uncertain components $\eta_1, \eta_2, \ldots, \eta_n$ with reliabilities $b_1, b_2, \ldots, b_n$, respectively. Since the structure function is

$$f(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = \left(\bigwedge_{i=1}^{m} \xi_i\right) \wedge \left(\bigwedge_{j=1}^{n} \eta_j\right),$$

we have

$$\begin{aligned}
Reliability &= \text{Ch}\left\{\left(\bigwedge_{i=1}^{m} \xi_i\right) \wedge \left(\bigwedge_{j=1}^{n} \eta_j\right) = 1\right\} \\
&= \text{Ch}\left\{\left(\bigwedge_{i=1}^{m} \xi_i = 1\right) \cap \left(\bigwedge_{j=1}^{n} \eta_j = 1\right)\right\} \\
&= \text{Pr}\left\{\bigcap_{i=1}^{m} (\xi_i = 1)\right\} \times \mathcal{M}\left\{\bigcap_{j=1}^{n} (\eta_j = 1)\right\} \\
&= \left(\prod_{i=1}^{m} \text{Pr}\{\xi_i = 1\}\right) \times \left(\bigwedge_{j=1}^{n} \mathcal{M}\{\eta_j = 1\}\right) \\
&= \left(\prod_{i=1}^{m} a_i\right) \cdot \left(\bigwedge_{j=1}^{n} b_j\right).
\end{aligned}$$

Remark 3.1 If the series system degenerates to a system containing only random components $\xi_1, \xi_2, \ldots, \xi_m$ with reliabilities $a_1, a_2, \ldots, a_m$, then

$$Reliability = \prod_{i=1}^{m} a_i.$$

If the series system degenerates to a system containing only uncertain components $\eta_1, \eta_2, \ldots, \eta_n$ with reliabilities $b_1, b_2, \ldots, b_n$, then

$$Reliability = \bigwedge_{j=1}^{n} b_j.$$

Example 3.2 (Parallel System) Consider a parallel system containing independent random components $\xi_1, \xi_2, \ldots, \xi_m$ with reliabilities $a_1, a_2, \ldots, a_m$, and independent uncertain components $\eta_1, \eta_2, \ldots, \eta_n$ with reliabilities $b_1, b_2, \ldots, b_n$, respectively. Since the structure function is

$$f(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = \left(\bigvee_{i=1}^{m} \xi_i\right) \vee \left(\bigvee_{j=1}^{n} \eta_j\right),$$

we have

$$\begin{aligned}
Reliability &= \mathrm{Ch}\left\{\left(\bigvee_{i=1}^{m} \xi_i\right) \vee \left(\bigvee_{j=1}^{n} \eta_j\right) = 1\right\} \\
&= 1 - \mathrm{Ch}\left\{\left(\bigvee_{i=1}^{m} \xi_i\right) \vee \left(\bigvee_{j=1}^{n} \eta_j\right) = 0\right\} \\
&= 1 - \mathrm{Ch}\left\{\left(\bigcap_{i=1}^{m} (\xi_i = 0)\right) \cap \left(\bigcap_{j=1}^{n} (\eta_j = 0)\right)\right\} \\
&= 1 - \mathrm{Pr}\left\{\bigcap_{i=1}^{m} (\xi_i = 0)\right\} \times \mathcal{M}\left\{\bigcap_{j=1}^{n} (\eta_j = 0)\right\} \\
&= 1 - \left(\prod_{i=1}^{m} \mathrm{Pr}\{\xi_i = 0\}\right) \times \left(\bigwedge_{j=1}^{n} \mathcal{M}\{\eta_j = 0\}\right) \\
&= 1 - \left(\prod_{i=1}^{m} (1 - a_i)\right) \cdot \left(\bigwedge_{j=1}^{n} (1 - b_j)\right).
\end{aligned}$$

Remark 3.2 If the parallel system degenerates to a system containing only random components $\xi_1, \xi_2, \ldots, \xi_m$ with reliabilities $a_1, a_2, \ldots, a_m$, then

$$Reliability = 1 - \prod_{i=1}^{m} (1 - a_i).$$

If the series system degenerates to a system containing only uncertain components $\eta_1, \eta_2, \ldots, \eta_n$ with reliabilities $b_1, b_2, \ldots, b_n$, then

$$Reliability = 1 - \left(\bigwedge_{j=1}^{n} (1 - b_j)\right) = \bigvee_{j=1}^{n} b_j.$$

4 Reliability index formula

This section aims at giving a formula to calculate the reliability of a system involving both random variables and uncertain variables.

Theorem 4.1 *Assume that a Boolean system has a structure function f and contains independent random components $\eta_1, \eta_2, \ldots, \eta_m$ with reliabilities $a_1, a_2, \ldots, a_m$, respectively, and independent uncertain components $\tau_1, \tau_2, \ldots, \tau_n$ with reliabilities $b_1, b_2, \ldots, b_n$, respectively. Then the reliability index of the uncertain random system is*

$$Reliability = \sum_{(y_1,\ldots,y_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(y_i)\right) \cdot Z(y_1, y_2, \ldots, y_m) \tag{3}$$

where

$$Z(y_1, \ldots, y_m) = \begin{cases} \sup\limits_{f(y_1,\ldots,y_m,z_1,\ldots,z_n)=1} \min\limits_{1\le j\le n} \nu_j(z_j), \\ \quad \text{if} \sup\limits_{f(y_1,\ldots,y_m,z_1,\ldots,z_n)=1} \min\limits_{1\le j\le n} \nu_j(z_j) < 0.5 \\ 1 - \sup\limits_{f(y_1,\ldots,y_m,z_1,\ldots,z_n)=0} \min\limits_{1\le j\le n} \nu_j(z_j), \\ \quad \text{if} \sup\limits_{f(y_1,\ldots,y_m,z_1,\ldots,z_n)=1} \min\limits_{1\le j\le n} \nu_j(z_j) \ge 0.5, \end{cases} \tag{4}$$

$$\mu_i(y_i) = \begin{cases} a_i & \text{if } y_i = 1 \\ 1 - a_i & \text{if } y_i = 0, \end{cases} \quad (i = 1, 2, \ldots, m), \tag{5}$$

$$\nu_j(z_j) = \begin{cases} b_j & \text{if } z_j = 1 \\ 1 - b_j & \text{if } z_j = 0 \end{cases} \quad (j = 1, 2, \ldots, n). \tag{6}$$

Proof It follows from Definition 3.1 of structure function and Definition 3.2 of reliability index that

$$Reliability = \text{Ch}\{f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n) = 1\}.$$

By the operational law of uncertain random variables (Theorem 2.5), we have

$$Reliability = \sum_{(y_1,\ldots,y_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(y_i)\right) \cdot \mathcal{M}\{f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n) = 1\}.$$

When $(y_1, \ldots, y_m)$ is given,

$$f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n) = 1$$

is a Boolean function of uncertain variables. It follows from the operational law of Boolean system (Theorem 2.2) that

$$\mathcal{M}\{f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n) = 1\} = Z(y_1, \ldots, y_m)$$

that is determined by (4), and we complete the proof.

Example 4.1 (k-out-of-n System) Consider a k-out-of-n system containing independent random components $\xi_1, \xi_2, \ldots, \xi_m$ with reliabilities $a_1, a_2, \ldots, a_m$, respectively, and independent uncertain components $\eta_1, \eta_2, \ldots, \eta_{n-m}$ with reliabilities $b_1, b_2, \ldots, b_{n-m}$, respectively. Note that the structure function is

$$f(y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{n-m}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{m} y_i + \sum_{j=1}^{n-m} z_j \geq k \\ 0, & \text{if } \sum_{i=1}^{m} y_i + \sum_{j=1}^{n-m} z_j < k. \end{cases}$$

It follows from Theorem 4.1 that the reliability of the uncertain random system is

$$Reliability = \sum_{(y_1,\ldots,y_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(y_i)\right) \cdot Z(y_1, y_2, \ldots, y_m)$$

in which

$$\mu_i(y_i) = \begin{cases} a_i & \text{if } y_i = 1 \\ 1 - a_i & \text{if } y_i = 0, \end{cases}$$

$$\begin{aligned} Z(y_1, y_2, \ldots, y_m) &= \mathcal{M}\left\{\sum_{i=1}^{m} y_i + \sum_{j=1}^{n-m} \eta_j \geq k\right\} \\ &= \begin{cases} \text{the } (k - \sum_{i=1}^{m} y_i)\text{th largest value of } b_1, b_2, \ldots, b_{n-m}, & \text{if } \sum_{i=1}^{m} y_i < k \\ 1, & \text{if } \sum_{i=1}^{m} y_i \geq k. \end{cases} \end{aligned}$$

Remark 4.1 If the k-out-of-n system degenerates to a system containing only random components $\xi_1, \xi_2, \ldots, \xi_n$ with reliabilities $a_1, a_2, \ldots, a_n$, respectively. Then

$$Reliability = \sum_{y_1+\cdots+y_n\geq k} \left(\prod_{i=1}^{n} \mu_i(y_i)\right).$$

If the k-out-n system degenerates to a system containing only uncertain components $\eta_1, \eta_2, \ldots, \eta_n$ with reliabilities $b_1, b_2, \ldots, b_n$, respectively. Then

$$Reliability = \text{the } k\text{th largest value of } b_1, b_2, \ldots, b_n.$$

Example 4.2 (Parallel–series system) Consider a simple parallel–series system in Fig. 1 containing independent random components ξ_1, ξ_2 with reliabilities a_1, a_2, respectively, and independent uncertain components η_1, η_2 with reliabilities b_1, b_2, respectively.

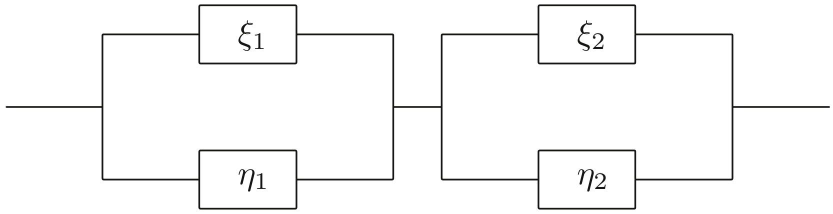
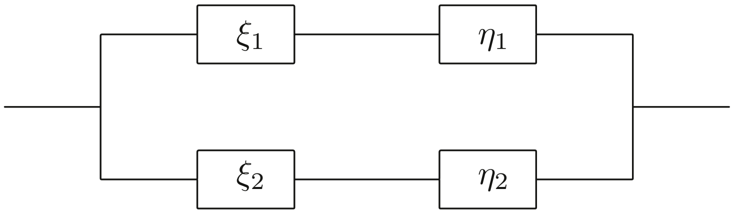


Fig. 1 Parallel–series system

Fig. 2 Series–parallel system



Note that the structure function is

$$f(\xi_1, \xi_2, \eta_1, \eta_2) = (\xi_1 \vee \eta_1) \wedge (\xi_2 \vee \eta_2).$$

It follows from Theorem 4.1 that the reliability index is

$$\begin{aligned} Reliability &= \text{Ch}\{(\xi_1 \vee \eta_1) \wedge (\xi_2 \vee \eta_2) = 1\} \\ &= \Pr\{\xi_1 = 1, \xi_2 = 1\} \cdot Z(1, 1) + \Pr\{\xi_1 = 1, \xi_2 = 0\} \cdot Z(1, 0) \\ &\quad + \Pr\{\xi_1 = 0, \xi_2 = 1\} \cdot Z(0, 1) + \Pr\{\xi_1 = 0, \xi_2 = 0\} \cdot Z(0, 0) \\ &= a_1 a_2 \cdot Z(1, 1) + a_1(1 - a_2) \cdot Z(1, 0) + (1 - a_1)a_2 \cdot Z(0, 1) \\ &\quad + (1 - a_1)(1 - a_2) \cdot Z(0, 0) \end{aligned}$$

where

$$\begin{aligned} Z(1, 1) &= \mathcal{M}\{(1 \vee \eta_1) \wedge (1 \vee \eta_2) = 1\} = \mathcal{M}\{1 \wedge 1 = 1\} = 1, \\ Z(1, 0) &= \mathcal{M}\{(1 \vee \eta_1) \wedge (0 \vee \eta_2) = 1\} = \mathcal{M}\{1 \wedge \eta_2 = 1\} = \mathcal{M}\{\eta_2 = 1\} = b_2, \\ Z(0, 1) &= \mathcal{M}\{(0 \vee \eta_1) \wedge (1 \vee \eta_2) = 1\} = \mathcal{M}\{\eta_1 \wedge 1 = 1\} = \mathcal{M}\{\eta_1 = 1\} = b_1, \\ Z(0, 0) &= \mathcal{M}\{(0 \vee \eta_1) \wedge (0 \vee \eta_2) = 1\} = \mathcal{M}\{\eta_1 \wedge \eta_2 = 1\} = b_1 \wedge b_2. \end{aligned}$$

Thus, the reliability index of the parallel–series system is

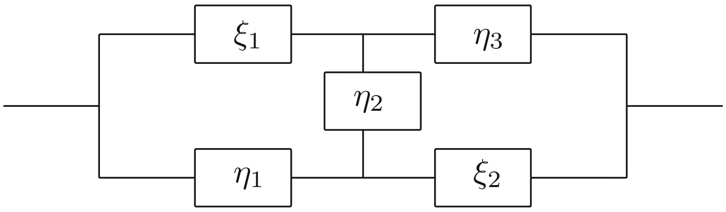
$$Reliability = a_1 a_2 + a_1(1 - a_2)b_2 + (1 - a_1)a_2 b_1 + (1 - a_1)(1 - a_2)(b_1 \wedge b_2).$$

Example 4.3 (Series–parallel system) Consider a simple series–parallel system in Fig. 2 containing independent random components ξ_1, ξ_2 with reliabilities a_1, a_2, respectively, and independent uncertain components η_1, η_2 with reliabilities b_1, b_2, respectively.

Note that the structure function is

$$f(\xi_1, \xi_2, \eta_1, \eta_2) = (\xi_1 \wedge \eta_1) \vee (\xi_2 \wedge \eta_2).$$

Fig. 3 Bridge system



It follows from Theorem 4.1 that the reliability index is

$$\begin{aligned}
Reliability &= \text{Ch}\{(\xi_1 \wedge \eta_1) \vee (\xi_2 \wedge \eta_2) = 1\} \\
&= \Pr\{\xi_1 = 1, \xi_2 = 1\} \cdot Z(1,1) + \Pr\{\xi_1 = 1, \xi_2 = 0\} \cdot Z(1,0) \\
&\quad + \Pr\{\xi_1 = 0, \xi_2 = 1\} \cdot Z(0,1) + \Pr\{\xi_1 = 0, \xi_2 = 0\} \cdot Z(0,0) \\
&= a_1 a_2 \cdot Z(1,1) + a_1(1-a_2) \cdot Z(1,0) + (1-a_1)a_2 \cdot Z(0,1) \\
&\quad + (1-a_1)(1-a_2) \cdot Z(0,0)
\end{aligned}$$

where

$$\begin{aligned}
Z(1,1) &= \mathcal{M}\{(1 \wedge \eta_1) \vee (1 \wedge \eta_2) = 1\} = \mathcal{M}\{\eta_1 \vee \eta_2 = 1\} = b_1 \vee b_2, \\
Z(1,0) &= \mathcal{M}\{(1 \wedge \eta_1) \vee (0 \wedge \eta_2) = 1\} = \mathcal{M}\{\eta_1 \vee 0 = 1\} = \mathcal{M}\{\eta_1 = 1\} = b_1, \\
Z(0,1) &= \mathcal{M}\{(0 \wedge \eta_1) \vee (1 \wedge \eta_2) = 1\} = \mathcal{M}\{0 \vee \eta_2 = 1\} = \mathcal{M}\{\eta_1 = 2\} = b_2, \\
Z(0,0) &= \mathcal{M}\{(0 \wedge \eta_1) \vee (0 \wedge \eta_2) = 1\} = \mathcal{M}\{0 \vee 0 = 1\} = 0.
\end{aligned}$$

Thus, the reliability index of the series–parallel system is

$$Reliability = a_1 a_2 (b_1 \vee b_2) + a_1(1-a_2) b_1 + (1-a_1) a_2 b_2.$$

Example 4.4 (Bridge System) Consider a simple bridge system in Fig. 3 containing independent random components ξ_1, ξ_2 with reliabilities a_1, a_2, respectively, and independent uncertain components η_1, η_2, η_3 with reliabilities b_1, b_2, b_3 respectively.

Note that the structure function is

$$f(\xi_1, \xi_2, \eta_1, \eta_2, \eta_3) = (\xi_1 \wedge \eta_3) \vee (\eta_1 \wedge \xi_2) \vee (\xi_1 \wedge \eta_2 \wedge \xi_2) \vee (\eta_1 \wedge \eta_2 \wedge \eta_3).$$

It follows from Theorem 4.1 that the reliability index is

$$\begin{aligned}
&Reliability \\
&= \text{Ch}\{(\xi_1 \wedge \eta_3) \vee (\eta_1 \wedge \xi_2) \vee (\xi_1 \wedge \eta_2 \wedge \xi_2) \vee (\eta_1 \wedge \eta_2 \wedge \eta_3)\} \\
&= \Pr\{\xi_1 = 1, \xi_2 = 1\} \cdot Z(1,1) + \Pr\{\xi_1 = 1, \xi_2 = 0\} \cdot Z(1,0) \\
&\quad + \Pr\{\xi_1 = 0, \xi_2 = 1\} \cdot Z(0,1) + \Pr\{\xi_1 = 0, \xi_2 = 0\} \cdot Z(0,0) \\
&= a_1 a_2 \cdot Z(1,1) + a_1(1-a_2) \cdot Z(1,0) + (1-a_1)a_2 \cdot Z(0,1) \\
&\quad + (1-a_1)(1-a_2) \cdot Z(0,0)
\end{aligned}$$

where

$$
\begin{aligned}
Z(1,1) &= \mathcal{M}\{(1 \wedge \eta_3) \vee (\eta_1 \wedge 1) \vee (1 \wedge \eta_2 \wedge 1) \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{\eta_3 \vee \eta_1 \vee \eta_2 \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{\eta_3 \vee \eta_1 \vee \eta_2 = 1\} \\
&= b_1 \vee b_2 \vee b_3, \\
Z(1,0) &= \mathcal{M}\{(1 \wedge \eta_3) \vee (\eta_1 \wedge 0) \vee (1 \wedge \eta_2 \wedge 0) \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{\eta_3 \vee 0 \vee 0 \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{\eta_3 = 1\} \\
&= b_3, \\
Z(0,1) &= \mathcal{M}\{(0 \wedge \eta_3) \vee (\eta_1 \wedge 1) \vee (0 \wedge \eta_2 \wedge 1) \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{0 \vee \eta_1 \vee 0 \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{\eta_1 = 1\} \\
&= b_1, \\
Z(0,0) &= \mathcal{M}\{(0 \wedge \eta_3) \vee (\eta_1 \wedge 0) \vee (0 \wedge \eta_2 \wedge 0) \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{0 \vee 0 \vee 0 \vee (\eta_1 \wedge \eta_2 \wedge \eta_3) = 1\} \\
&= \mathcal{M}\{\eta_1 \wedge \eta_2 \wedge \eta_3 = 1\} \\
&= b_1 \wedge b_2 \wedge b_3.
\end{aligned}
$$

Thus, the reliability index of the series–parallel system is

$$
\begin{aligned}
Reliability = a_1 a_2 (b_1 \vee b_2 \vee b_3) + a_1(1-a_2)b_3 + (1-a_1)a_2 b_1 \\
+ (1-a_1)(1-a_2)(b_1 \wedge b_2 \wedge b_3).
\end{aligned}
$$

5 Conclusion

This paper mainly proposed the concept of reliability index in uncertain random systems. A reliability index theorem was derived to calculate the reliability index. Moreover, some special common systems in uncertain random environment such as k-out-of-n system, parallel–series system, series–parallel system and bridge system were discussed.

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REVIEW ARTICLE

Measuring reliability under epistemic uncertainty: Review on non-probabilistic reliability metrics

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Uncertainty theory

Abstract In this paper, a systematic review of non-probabilistic reliability metrics is conducted to assist the selection of appropriate reliability metrics to model the influence of epistemic uncertainty. Five frequently used non-probabilistic reliability metrics are critically reviewed, i.e., evidence-theory-based reliability metrics, interval-analysis-based reliability metrics, fuzzy-interval-analysis-based reliability metrics, possibility-theory-based reliability metrics (posbist reliability) and uncertainty-theory-based reliability metrics (belief reliability). It is pointed out that a qualified reliability metric that is able to consider the effect of epistemic uncertainty needs to (1) compensate the conservatism in the estimations of the component-level reliability metrics caused by epistemic uncertainty, and (2) satisfy the duality axiom, otherwise it might lead to paradoxical and confusing results in engineering applications. The five commonly used non-probabilistic reliability metrics are compared in terms of these two properties, and the comparison can serve as a basis for the selection of the appropriate reliability metrics.

1. Introduction

Reliability refers to the capacity of a component or a system to perform its required functions under stated operating conditions for a specified period of time.[1] Reliability engineering has nowadays become an independent engineering discipline, which measures the reliability by quantitative metrics and controls it via reliability-related engineering activities implemented in the product lifecycle, i.e., failure mode, effect and criticality analysis (FMECA),[2] fault tree analysis (FTA),[3] environmental stress screening (ESS),[4] reliability growth testing (RGT),[5] etc. Among all the reliability-related engineering activities, measuring reliability is a fundamental one.[6] Measuring reliability

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refers to quantifying the reliability of a component or system by quantitative metrics. A key problem in measuring reliability is how to deal with the uncertainty affecting the product's reliability. Broadly speaking, uncertainty can be categorized as aleatory uncertainty which refers to the uncertainty inherent in the physical behavior of the system,[7,8] and epistemic uncertainty which refers to the uncertainty that is caused by incomplete knowledge.[7,9]

In the early years of reliability engineering, reliability has been measured by probability-based metrics, e.g., in terms of the probability that the component or system does not fail (referred to as probabilistic reliability in this paper[10]), and estimated by statistical methods based on failure data (e.g., see Ref.[11]). However, in engineering practice, the available failure data, if there are any, are often far from sufficient for accurate statistical estimates.[12] Also, the statistical methods do not explicitly model the actual process that leads to the failure. Rather, the failure process is regarded as a black box and assumed to be uncertain, which is described indirectly based on the observed distribution of the time-to-failure (TTF). From the perspective of uncertainties, the statistical methods do not separate the root causes of failures and uncertainties and therefore, they do not distinguish between aleatory and epistemic uncertainties.

As technology evolves, modern products often have high reliability, making it even harder to collect enough failure data, which severely challenges the use of statistical methods.[13] At the same time, as the knowledge of the failure mechanisms accumulates, deterministic models are available to describe the failure process based on the physical knowledge of the failure mechanisms (referred to as physics-of-failure (PoF) models[14]). An alternative method to estimate the probabilistic reliability is, then, developed based on the PoF models. In this paper, these methods are referred to as the model-based methods. Unlike statistical methods, model-based methods treat the actual failure process as a white box: the TTFs are predicted by deterministic PoF models, while the uncertainty affecting the TTF is assumed to be caused by random variations in the model parameters (aleatory uncertainty). The probabilistic reliability is, then, estimated by propagating aleatory uncertainties through the model analytically or numerically, e.g., by Monte Carlo simulation.[15,16] Compared to statistical methods, model-based methods explicitly describe the actual failure process (by the deterministic PoF models) and separate the root cause of failures (assumed to be deterministic) and the aleatory uncertainty (the random variation of model parameters). The separation of deterministic root causes and aleatory uncertainty allows the designer to implement parametric design for reliability, e.g., the reliability-based design optimization (RBDO),[17,18] tolerance optimization,[19,20] etc., which marks significant advancement in reliability engineering.

From the perspective of uncertainties, only aleatory uncertainty is considered in the model-based methods. In practice, however, the trustfulness of the predicted reliability is severely influenced by epistemic uncertainty. As in today's highly competitive markets, it is more and more frequent to use the model-based method to measure reliability, due to the severe shortage on failure data. To better quantify the reliability with the model-based methods, the effect of epistemic uncertainty should also be considered. Epistemic uncertainty relates to the completeness and accuracy of the knowledge: if the failure process is poorly understood, there will be large epistemic uncertainty.[21–23] For instance, the deterministic PoF model might not be able to perfectly describe the failure process, e.g., due to incomplete understanding of the failure causes and mechanisms.[21,24] Besides, the precise values of the model parameters might not be accurately estimated due to lack of data in the actual operational and environmental conditions. Both of these two factors introduce epistemic uncertainty into the reliability estimation: the more severe the effect of these factors is, the less trustful the predicted reliability is.

In literature, there are various approaches to measure reliability under epistemic uncertainty, e.g., probability theory (subjective interpretation[25,26]), evidence theory,[27] interval analysis,[28,29] fuzzy interval analysis,[30] possibility theory,[31,32] uncertainty theory,[33] etc. In this paper, a critical review on these reliability metrics is conducted to assist the selection of appropriate metrics. Some researchers and practitioners use probability theory to describe epistemic uncertainty, taking a Bayesian interpretation of probability.[25,26] In recent years, problems in dealing with epistemic uncertainty by probabilistic methods have been pointed out.[34,35] Non-probabilistic metrics have, then, been proposed to model epistemic uncertainty. In this paper, we discuss these non-probabilistic reliability metrics.

More specifically, five reliability metrics are discussed in this paper, i.e., evidence-theory-based reliability metrics, interval-analysis-based reliability metrics, fuzzy-interval-analysis-based reliability metrics, possibility-theory-based reliability metrics (posbist reliability) and uncertainty-theory-based reliability metrics (belief reliability). They are classified, based on the mathematical essence of the metrics, as probability-interval-based and monotone-measure-based reliability metrics. The former refers to an interval that contains all the possible reliabilities/failure probabilities, while the latter refers to reliability metrics that are defined based on a monotone measure (or fuzzy measure[36]). A further classification is given in Fig. 1. The probability-interval-based and monotone-measure-based reliability metrics are reviewed in Sections 2 and 3, respectively.

2. Probability-interval-based reliability metrics

Probability-interval-based reliability metrics (PIB metrics) describe the effect of epistemic uncertainty by an interval of values of failure probabilities/reliabilities. The width of the interval represents the extent of epistemic uncertainty: wide intervals represent large epistemic uncertainty. When there is no effect of epistemic uncertainty, the probability interval becomes a single distribution function of the TTFs. We consider three of the most popular non-probabilistic methods for epistemic uncertainty representation, i.e., evidence theory, interval analysis (probability box) and fuzzy interval analysis. We review each of these three methods separately in the remaining of this section.

2.1. Evidence-theory-based methods

Evidence theory, also known as Dempster–Shafer theory or as the theory of belief functions, was established by Shafer[37] for representing and reasoning with uncertain, imprecise and incomplete information.[38] It is a generalization of the Bayesian theory of subjective probability in the sense that it does not require probabilities for each event of interest, but bases the

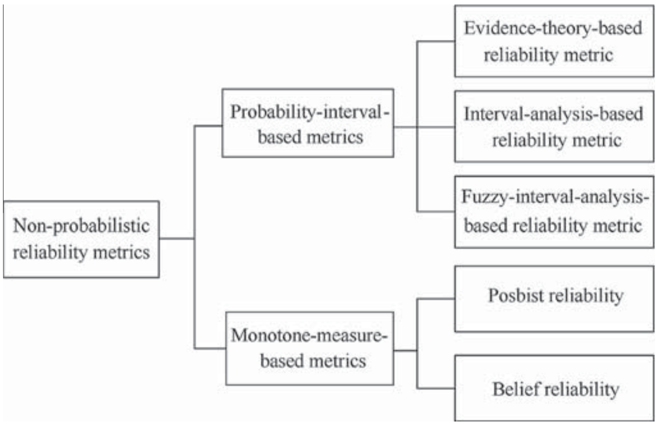

Fig. 1 Classification of existing non-probabilistic reliability metrics.

belief in the truth of an event on the probabilities of other propositions or events related to it.[37] Evidence theory provides an alternative to the traditional manner in which probability theory is used to represent uncertainty by means of the specification of two degrees of likelihood, belief and plausibility, for each event under consideration. The belief value of an event measures the degree of belief that the event will occur and the plausibility value measures the extent to which evidence does not support the negation of the event. Evidence theory is applied to describing uncertainty when the application of probability theory cannot be supported, e.g., when few samples of data are available to estimate the probability accurately.[37]

To obtain the evidence-theory-based reliability metrics, the first step is to define the frame of discernment:

$$\Theta = \{\theta_1, \theta_2, ..., \theta_m\} \quad (1)$$

where the set Θ includes all the possible and mutually exclusive elementary propositions or hypotheses with respect to the uncertain events. Let A_i (i = 1, 2, ..., 2^m) denote the subsets of Θ. All the subsets (also called focal sets) compose the power set of Θ, which is denoted by 2^Θ. Next, basic probability assignment (BPA) is assigned to each focal set to represent our belief in the event associated to it. BPA is essentially a mapping function $m:2^\Theta \rightarrow [0, 1]$, which satisfies

(1) $m(\varnothing) = 0$
(2) $\sum_{A_i \subseteq \Theta} m(A_i) = 1$

In practice, the values of the BPAs are assigned by experts to represent the effect of epistemic uncertainty. Focal sets and their associated BPAs comprise the evidence, based on which the belief and plausibility of an event B can be calculated:

$$\begin{cases} \mathrm{Bel}(B) = \sum_{A_i \subseteq B} m(A_i) \\ \mathrm{Pl}(B) = \sum_{A_i \cap B \neq \varnothing} m(A_i) \end{cases} \quad (2)$$

where A_i denote the focal sets and $m(A_i)$ is its BPA.

The belief in event B is quantified as the sum of the masses assigned to all sets enclosed by it; hence, it can be interpreted as a lower bound representing the amount of belief that supports the event. The plausibility of event B is, instead, the sum of the BPAs assigned to all sets whose intersection with event B is not empty; hence, it is an upper bound on the probability that the event occurs.[39] Thus,

$$\mathrm{Bel}(B) \leqslant P(B) \leqslant \mathrm{Pl}(B) \quad (3)$$

When the event B is the failure of a component or system, Eq. (3) leads to an interval that contains all possible failure probabilities/reliabilities, representing the effect of epistemic uncertainty on the reliability estimation: the larger the width of the interval, the greater the epistemic uncertainty is, and thus, the less we can trust the estimated reliability.

Rakowsky reviewed some early applications of evidence-theory-based reliability metrics constructed based on failure modes and effects analysis (FMEA), event tree analysis (ETA) and FTA.[40] Mourelatos and Zhou used evidence theory to construct failure probability intervals and applied them in engineering design optimization.[41–43] In reliability-based optimization (RBO), based on the interval of failure probability, Alyanak et al. developed a new method for projecting gradients in RBO when available data are not enough.[44] Yao et al. developed a sequential optimization and mixed uncertainty analysis method for RBO, where evidence theory is used to describe epistemic uncertainty.[45] Similar to Bayesian network, the evidential network was developed to construct the failure probability intervals.[46] Yang et al. applied the evidential network to FTA and calculated the failure probability intervals.[47] Bae et al. constructed failure probability intervals in large-scale structures based on evidence theory by identifying the failure region and expressing it as a function of the focus sets.[27,48] Considering the large computing cost, Bae et al. introduced an approximation method to calculate the failure probability intervals under the framework of evidence theory.[49] Jiang et al. developed an efficient evaluation method for structure reliability with epistemic uncertainty using evidence theory, which reduced the computation cost compared with traditional methods.[50] To solve the problem of constructing failure intervals with dependent parameters, Jiang et al. developed a multidimensional evidence-theory model, where the dependency is addressed by an ellipsoidal model.[51] Baraldi et al. studied the situation in which a number of experts provided different information about the imprecise parameters, and belief and plausibility functions are used to develop upper and lower bounds of cumulative probability functions.[52,53] Lo et al. assessed seismic probabilistic risk of nuclear power plants and built associated failure probability intervals based on evidence theory.[54] Khalaj et al. applied evidence theory to risk-based reliability analysis.[55] Yao et al. studied the uncertainty quantification in multidisciplinary optimization and developed a new method to calculate the failure probability intervals based on optimization in the framework of evidence theory.[56,57]

2.2. Interval-analysis-based methods

Another way to construct the interval of failure probabilities is to use interval analysis (or probability boxes). Given a model $y = f(x)$, interval analysis assumes that the input variable x is subjected to epistemic uncertainty and is described by an interval (or convex sets if the input variables are multidimensional) comprised of an lower bound x_L and an upper bound x_U, so that $x_L \leqslant x \leqslant x_U$. Then, interval mathematics or numerical optimization methods are used to derive the upper and lower bounds of the output variable y.[58] When interval analysis is applied to probabilistic models, upper and lower bounds of the probability of interests can be calculated, which form a probability "box" (p-box) that contains all possible values of that probability. Since reliability is calculated by a probabilistic model, the p-box becomes a natural tool to describe the epistemic uncertainty influencing the calculated reliability.

Ferson et al. are among the first ones who apply the p-box to describing and propagating epistemic uncertainty in a reliability model, deriving intervals that contain all possible values of failure probabilities.[59,60] Karanki et al. applied p-box to evaluate the probability of system failure under the influence of epistemic uncertainty.[61] Using a similar method to describe epistemic uncertainty, Zhang et al. developed interval Monte Carlo simulation methods,[62] interval importance sampling methods[63] and quasi-Monte Carlo methods[64] to calculate the interval of failure probabilities when the structures are implicitly modeled based on a finite element model. Beer et al. developed a calculation method for failure probability intervals, which is specially designed for small sample size and is based on quasi-Monte Carlo simulations.[65,66] Xiao et al. put forward a saddle-point-based approximation method to enhance the computational efficiency in calculating the interval of structural failure probability.[67] Qiu et al. developed methods to construct the interval of failure probabilities with small sample size, using numerical optimization methods.[68–70] Crespo et al. applied p-box to the analysis of polynomial systems subject to parameter uncertainties.[71]

2.3. Fuzzy-interval-analysis-based method

Fuzzy-interval-analysis-based method allows the consideration of both aleatory and epistemic uncertainty simultaneously.[34] The method can be regarded as the combination of probability theory and fuzzy set theory, where the effect of aleatory uncertainty is described by probability distributions, while the effect of epistemic uncertainty is described by possibility distributions. For instance, in a model $z = f(x, y)$, the input variable x might be subject to aleatory uncertainty and described by a probability density function $f_X(\cdot)$; while the other variable y might be subject to epistemic uncertainty and described by a possibility distribution $\Pi_y(\cdot)$ (often through expert opinion elicitations).

Kaufmann and Gupta introduced the basic idea of expressing randomness (probability) in combination with imprecision (possibility) via hybrid numbers.[72] Ferson et al.[73,74] extended Kaufmann's work by developing computational rules of hybrid numbers (i.e., the probability distributions are fuzzily known), which can be applied in risk assessment. Through the computational method, the random fuzzy sets can be obtained and converted to the upper and lower bounds of failure probability. Guyonnet et al. introduced a hybrid method to propagate both aleatory and epistemic uncertainties using fuzzy interval analysis.[75] In this method, the possibility distribution function of the output variable z can be first calculated based on the Monte-Carlo sampling method and the possibility extension principle, and then used to derive the upper and lower bounds of failure probabilities based on fuzzy interval analysis.[76] Baudrit et al. developed a postprocessing method based on belief functions (evidence theory) to extract useful information and to construct the failure probability bounds based on the results of the hybrid method,[34] and they proved that the method improved the work of Ferson et al.[73,74] and Guyonnet et al.[75] Baraldi and Zio summarized the hybrid method that jointly propagates probabilistic and possibilistic uncertainties, and compared the method with pure probabilistic and pure fuzzy methods.[77] Based on the work of Baudrit et al.[34] Li and Zio applied the fuzzy interval analysis method to assess the reliability of a distributed generation system, which is affected by serious epistemic uncertainty.[30] The hybrid fuzzy interval analysis method has also been applied successfully in other areas, e.g., reliability assessment of a flood protection dike[78] and a turbo-pump lubricating system.[79] Flage et al. used probabilistic-possibilistic computational framework to propagate uncertainties in FTA, giving rise to the failure probability bounds of top event.[80] Li et al. developed a hybrid-universal-generating-function-based (HUGF) method for the fuzzy interval analysis of multi-state systems.[81]

2.4. Problem with PIB metrics

Although differences exist in the way that the interval of failure probabilities is constructed, all the three methods reviewed in Sections 2.1–2.3 use this interval as the reliability metrics. The width of the interval reflects the extent of epistemic uncertainty. One important problem in reliability theory is how to calculate the system-level reliability metrics based on the reliability metrics of the components. Since PIB metrics are intervals of probabilities, the system-level PIB metrics are calculated based on the laws of probability theory. This fact causes a common problem for the PIB metrics when applied to calculate system reliability metrics. Consider the following example.

Example 1. Consider a series system composed of 30 components. Suppose that the real reliability of each component is 0.95. Since the system is subject to epistemic uncertainty, the PIB metrics are used to quantify the reliability of the components. We suppose that the reliability interval for each component is [0.9, 1]. Then, following the laws of probability theory, the system's PIB reliability metric will be $[0.9^{30}, 1^{30}] = [0.04, 1]$. This interval is not representative of the actual uncertainty on the system reliability and obviously too wide to provide any valuable information in practical applications.

The reason for the unsatisfactory result in Example 1 is that the imprecision in the component reliability metrics (the width of the interval) is amplified by the product law of probability theory that calculates the intersection of events. The system-level reliability metric should be able to compensate for the conservatism in the component-level reliability metrics caused by the consideration of epistemic uncertainty. Monotone-measure-based reliability metrics are developed for this aim.

3. Monotone-measure-based reliability metrics

Monotone measure was defined by Choquet as a generalization of the classical measure theory.[82] Let X be a finite universal set, and let l be a non-empty family of subsets of X. Then g: $l \rightarrow [0, \infty]$ is a monotone measure on (X, l) if it satisfies the following requirements:

(1) $g(\varnothing) = 0$;
(2) $\forall A, B \in l$, if $A \subseteq B$, then $g(A) \leqslant g(B)$.

Probability measure is a special case of the monotone measure, which is also additive. As pointed out by Klir and Smith,[83] non-additive monotone measures might be able to represent broader types of uncertainty than the addictive probability theory. Therefore, they are applied to developing reliability metrics that model epistemic uncertainty. Typical monotone-measure-based reliability metrics include posbist reliability which is based on possibility theory, and belief reliability which is based on uncertainty theory.

3.1. Possibility-theory-based reliability metrics

The most widely applied possibility-theory-based reliability metric is the posbist reliability. The two basic assumptions of posbist reliability are[32,84]

(1) Possibility assumption: the system failure behavior is fully characterized in the context of possibility measures.
(2) Binary-state assumption: the system demonstrates only two crisp states, i.e. fully functioning or fully failed. At any time, the system is in one of the two states.

In posbist reliability theory, lifetime of a system (or a component) is a non-negative real-valued fuzzy variable, and the posbist reliability of a system (or a component) is defined as the possibility measure that the system (or the component) performs its assigned functions properly during a predefined exposure period in a given environment.[84] The epistemic uncertainty is, then, described and propagated based on possibility theory.

Following the definition of posbist reliability, Cai et al. developed posbist reliability analysis methods for series, parallel, series–parallel, parallel–series and coherent systems.[84,85] Huang et al. proposed detailed posbist reliability analysis methods for k-out-of-n: G systems.[86] Cai et al. studied posbist reliability behavior of cold stand-by and warm stand-by systems, considering both full reliable and non-full reliable conversion switches.[87] Utkin et al. extended Cai's work to repairable systems and developed a posbist reliability analysis method based on state transition diagram.[88,89] Huang et al. introduced a posbist reliability fault tree analysis (posbist FTA) method for coherent systems to evaluate reliability and safety.[90] He et al. developed calculation methods of posbist reliability for typical systems when the components are symmetric Gaussian fuzzy variables.[91] Bhattacharjee et al. investigated the posbist reliability of k-out-of-n systems and pointed out that the posbist reliability does not depend on the number of components.[92]

In essence, posbist reliability is a possibility measure. In possibility theory, the possibility measure $\Pi(\cdot)$ satisfies the following three axioms:[93]

Axiom 1. For the empty set $\varnothing$, there is $\Pi(\varnothing) = 0$.

Axiom 2. For the universal set Γ, there is $\Pi(\Gamma) = 1$.

Axiom 3. For any events Λ_1 and Λ_2 in the universal set Γ, there is $\Pi(\Lambda_1 \cup \Lambda_2) = \max(\Pi(\Lambda_1), \Pi(\Lambda_2))$.

Axiom 3 shows that the operation laws of possibility theory differ from those of probability theory. Therefore, the system reliability analysis method is also different from that based on probability theory. For instance, Cai et al. proved that the system posbist reliability is the minimum one among all the posbist reliabilities of its components.[32] This difference makes it possible for possibility theory to compensate the conservatism caused by epistemic uncertainty in component-level reliability estimations.

Example 2. Consider a series system composed of 300 components. An extreme case is considered where all the components are designed with sufficient margins, so that they are completely reliable and the real reliability should be 1. It is easy to verify that the system's reliability is also 1, which means that the system is highly reliable. However, since the system is subject to epistemic uncertainty, the estimates of component-level reliabilities are likely to be conservative. We suppose, for example, the reliability of each component is estimated to be $R_1 = R_2 = \cdots = R_{300} = 0.99$. If we use probability theory to model the reliability metric, the system reliability is

$$R_S = R_1 R_2 \cdots R_{300} = 0.04$$

It can be seen from the result that the conservatism in component-level reliability estimates is amplified by the operation laws of probability theory, which contradicts with our intuitions since a highly reliable system is judged as highly unreliable.

If we use the posbist reliability, however, the system reliability is

$$R_S = \min(R_1, R_2, \cdots, R_{300}) = 0.99$$

which avoids the previous counter-intuitive result and demonstrates that possibility theory can compensate the conservatism in the component-level reliability estimates caused by epistemic uncertainty.

3.1.1. Problems with posbist reliability

A major drawback of the possibility-theory-based reliability metrics is that the possibility measure does not satisfy the duality axiom, which might lead to counter-intuitive results when applied in practical reliability-related applications.

Example 3. Let event $\Lambda_1 = \{\text{The system is working}\}$ and $\Lambda_2 = \{\text{The system fails}\}$. It is obvious that the universal set $\Gamma = \Lambda_1 \cup \Lambda_2$. Also, we have the posbist reliability and posbist unreliability to be $R_{\text{pos}} = \Pi(\Lambda_1)$ and $\overline{R_{\text{pos}}} = \Pi(\Lambda_2)$, respectively. According to Axioms 2 and 3, we have

$$\Pi(\Gamma) = \Pi(\Lambda_1 \cup \Lambda_2) = \max(R_{\text{pos}}, \overline{R_{\text{pos}}}) = 1 \tag{4}$$

Therefore, if R_{pos} does not equal to 1, e.g., $R_{\text{pos}} = 0.8$, $\overline{R_{\text{pos}}}$ must equal to 1. Vice versa, if $\overline{R_{\text{pos}}}$ does not equal to 1, e.g., $\overline{R_{\text{pos}}} = 0.8$, R_{pos} must equal to 1. This is a counterintuitive result and easily confuses the decision maker in real applications. Hence, even though designed to consider epistemic uncertainty, a reliability metric should still satisfy the duality axiom.

3.2. Uncertainty-theory-based reliability metrics

As just explained in Section 3.1.1, one major drawback of the possibility-theory-based reliability metrics is that possibility theory does not satisfy the duality axiom. To overcome this drawback, belief reliability has been developed based on uncertainty theory. Founded by Liu,[33,94] uncertainty theory relies on the uncertain measure to describe the belief degree of events affected by epistemic uncertainty, which is a monotone measure based on the following four axioms:

1) Normality axiom: $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
2) Duality axiom: $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
3) Subadditivity axiom: for every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leqslant \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$;
4) Product axiom: Let $(\Gamma_k, L_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertainty measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}$, where L_k are σ -algebras over Γ_k, and Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \ldots$, respectively.

Belief reliability was defined by Zeng et al. as the uncertainty measure of the system to perform specified functions within given time under given operating conditions.[95] Zeng et al. developed an evaluation method for component belief reliability, which incorporates the influences from design margin, aleatory uncertainty and epistemic uncertainty.[96] The issue of quantifying the effect of epistemic uncertainty is addressed by developing a method based on the performance of engineering activities related to reducing epistemic uncertainty.[97,98] The reason why uncertainty theory should be chosen as the theoretical foundation of belief reliability was explained by Zeng et al.[99] by comparing it with other commonly encountered theories to deal with epistemic uncertainty, i.e., evidence theory, possibility theory, Bayesian theory, etc. system reliability analysis methods are also developed for coherent systems.[95,99]

Compared to the PIB metrics, belief reliability uses the minimum operation to calculate the belief degree of the intersection events, and therefore can compensate for the conservatism in the component-level reliability metrics caused by the consideration of epistemic uncertainty. Compared to the possibility-theory-based reliability metrics, belief reliability satisfies the duality axiom, which avoids the possible paradoxical results often encountered in engineering applications of the possibility-theory-based reliability metrics. Therefore, belief reliability is a promising reliability metric to measure the reliability affected by epistemic uncertainty. However, the researches in the theory of belief reliability are far from mature. In fact, as shown in the classical probability-based reliability theory, there are four major topics in the research of reliability theory:

(1) How to measure reliability (measurement).
(2) How to evaluate the reliability of a system based on the reliability of its components (analysis).
(3) How to design the system so that the desired reliability level can be fulfilled (design).

Table 1 Comparison of five reliability metrics.

Non-probabilistic metrics		Theory basis	Representative literature	Method to obtain metric	Existing problems
PIB reliability metrics	Evidence-theory-based reliability metric	Evidence theory	42	Use belief and plausibility functions to express the lower and upper bounds of failure probability.	The metrics are not able to compensate the conservatism in the estimated component-level reliability metrics, arising from the consideration of epistemic uncertainty.
	Interval-analysis-based reliability metric	Interval analysis	59, 61	Calculate the maximum and minimum of failure probability through interval analysis, given the range of input parameters.	
	Fuzzy-interval-analysis-based reliability metric	Fuzzy interval analysis	30, 34	First establish the possibility distribution of failure probability through Monte Carlo simulation and fuzzy interval analysis, and then obtain the bounds of failure probability via evidence theory.	
Monotone-measure-based reliability metrics	Posbist reliability	Possibility theory	85	Use possibility measure to calculate products' reliability.	The metric does not satisfy duality axiom.
	Belief reliability	Uncertainty theory	95	Obtain the belief reliability through calculating products' design margin, aleatory uncertainty factor and epistemic uncertainty factor.	The research is far from mature.

(4) How to demonstrate that the system satisfies its reliability requirements (demonstration).

Among the four topics, measurement is the most fundamental one. Since belief reliability is an entirely different reliability metric from the classical probability-based reliability metrics, new analysis, design and demonstration methods are also needed for the theory of belief reliability. As reviewed before, however, current researches on belief reliability only concentrate on the first two problems. The problems of design and demonstration are still relatively unexplored and deserve further investigations.

To summarize, we make a comparison of the five reviewed reliability metrics (see Table 1) in terms of theory basis, methods to obtain metric, and existing problems. This will help people to choose appropriate reliability metric according to different demands and situations.

4. Conclusions

In this paper, a systematic review is conducted on the non-probabilistic reliability metrics that are used to describe the effect of epistemic uncertainty. Five reliability metrics are discussed, i.e., the evidence-theory-based, interval-analysis-based, fuzzy-interval-analysis-based, possibility-theory-based (posbist reliability) and uncertainty-theory-based reliability metrics (belief reliability). Among them, the former three provide, in essence, an interval that contains all the possible values of the reliabilities/failure probabilities whereas the latter two give monotone measures.

An investigation of the five metrics reveals two important features that a qualified reliability metric under epistemic uncertainty should possess: (1) it should be able to compensate the conservatism in the component-level reliability metrics caused by the consideration of epistemic uncertainty, and (2) it should satisfy the duality axiom, otherwise it might lead to paradoxical and confusing results in engineering applications.

Finally, the five reliability metrics are compared with respect to the above two features, as well as other important characteristics which can be used to assist the selection of appropriate reliability metrics considering the effect of epistemic uncertainty.

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METHODOLOGIES AND APPLICATION

Sensitivity and stability analysis of the additive model in uncertain data envelopment analysis

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Abstract DEA is a non-parametric productive efficiency measurement method for operations with multiple inputs and multiple outputs. An issue which has received widespread attention in rapidly growing field of DEA is the sensitivity and stability of the evaluating results to perturbations in the data. Due to the uncertainty of the data in real life, this paper will perform a sensitivity and stability analysis of the additive model with uncertain inputs and output. Some theories of the uncertain DEA model based on uncertainty theory will be given. It is followed by some computation methods of the stable regions to the efficient and inefficient DMUs. Finally, a numerical example will be presented to give an illustration of the sensitivity and stability analysis.

Keywords Data envelopment analysis · Uncertain measure · Uncertainty distribution · Efficiency · Sensitivity · Stability

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1 Introduction

Data envelopment analysis (DEA), which was originated by Charnes et al. (1978) in 1978, is a mathematical method for determining the relative efficiency of decision-making units (DMUs) with multiple input and output. This was followed by a variety of theoretical research work, including those by Charnes et al. (1985), Petersen (1990), Tone (2001), and so on. In about 30 years, DEA has grown into a powerful analytical tool for evaluating system performance and has been successfully applied to a host of many different types of entities engaged in a wide variety of activities in many contexts worldwide. However, uncertainty, such as a measurement error, may be incorporated in the observed data. This indicates the necessity to assess the sensitivity of classifications in DEA.

The topic of sensitivity and stability analysis has taken a variety of forms in the DEA literature. Research on analytical approaches to sensitivity analysis in DEA was initiated in Charnes et al. (1985), which examined the change in a single input or output. This work was extended and improved in a series of papers (Charnes and Neralic 1992; Neralic 1997). Another avenue for sensitivity analysis was provided by Zhu (1996) and Seiford and Zhu (1998), who obtained the largest stability region when inputs or outputs changed individually. The two approaches described above use DEA “envelopment models” to treat one DMU at a time. Extensions are needed if all data vary simultaneously until the status of at least one DMU is changed. An approach initiated in Thompson et al. (1994) moves in this direction in that manner. Seiford and Zhu (1999) generalized the technique in Zhu (1996) and Seiford and Zhu (1998) to the case, where the efficiency of the underevaluation efficient DMU deteriorates while the efficiencies of the other DMUs improve.

The original DEA models assume that inputs and outputs are measured by exact values. However, in many situations, inputs and outputs are volatile and complex, so that they are difficult to measure in an accurate way. Thus, some researchers employed probability theory to establish some stochastic DEA models. Sengupta (1982) generalized the stochastic DEA model by using the expected value to the stochastic inputs and outputs. Banker (1993) incorporated statistical elements into DEA and developed an approach which effects inferences in statistical noise. Many papers (Olesen and Petersen 1995; Banker 1986; Cooper et al. 1996) have introduced chance-constrained programming into DEA to accommodate stochastic variations in data. Recently, many researchers have addressed this problem with fuzzy data. We can find several fuzzy approaches to the assessment of efficiency in the DEA literature, e.g., Kao and Liu (2000), Guo and Tanaka (2001) and Lertworasirikul et al. (2003).

In 2007, Liu (2007) founded an uncertainty theory to deal with human's belief degree mathematically and refined it in 2010 (Liu 2010). As we know, to obtain the probability distribution, we need a lot of samples. However, due to economical or technological reasons, sometimes we have no samples. In this case, we have to invite the domain experts to evaluate the belief degree that each possible event happens. Since humans tend to overweigh unlikely events (Kahneman and Tversky 1979), the belief degree has a much larger variance than the frequency and cannot be treated as a probability distribution of a random variable. In this case, we can regard the belief degree as an uncertainty distribution of some uncertain variable and deal with it via uncertainty theory. Uncertain programming, as a spectrum of mathematical programming involving uncertain variables, was proposed by Liu (2009) in 2009. Then, uncertain multilevel programming (Liu and Yao 2014), as well as uncertain multiobjective programming (Liu and Chen 2014), has been further studied. Since then, uncertainty theory has been used to solve a variety of real optimal problems, including facility allocation problem (Gao 2012), reliability analysis (Zeng et al. 2013) and so on. Since Wen et al. (2014) have proposed a new DEA model with uncertain inputs and outputs, this paper will discuss the sensitivity and stability in DEA with uncertain data.

Cognitive analysis which has been developed for many years in research on semantic data analysis (Duda et al. 2001) is another approach to deal with the uncertain data. Cognitive resonance was used for cognitive data analysis systems by Ogiela and Ogiela (2011). Tadeusiewicz et al. (2006) have applied cognitive analysis to business planning and decision support systems.

The remainder of this paper is organized as follows. Section 2 will introduce some basic concepts and properties about uncertain variables. Some introduction to uncertain DEA model proposed by Wen et al. (2014) is given in Sect. 3. Based on some theories, the stable regions of the inefficient DMUs will be given in Sect. 4. Section 5 will propose some new uncertain DEA models and some new theories to give the stable regions of the efficient DMUs. In Sect. 6, the computation methods of the stable regions are proposed. Finally, the analysis of the sensitivity and stability is introduced through a numerical example in Sect. 7.

2 Preliminaries

The uncertainty theory was founded by Liu (2007) in 2007 and refined by Liu (2010) in 2010. Nowadays, uncertainty theory has become a branch of axiomatic mathematics for modeling human uncertainty. In this section, we will state some basic concepts and results on uncertain variables. These results are crucial for the remainder of this paper.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\} \in [0, 1]$. To ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu (2007) presented the four axioms:

Axiom 1. $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3. For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$, then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

If the set function $\mathcal{M}$ satisfies the first three axioms, it is called an uncertain measure.

Definition 1 (Liu 2007) Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2 (Liu 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma\}|\xi(\gamma) \in B\} \quad (1)$$

is an event.

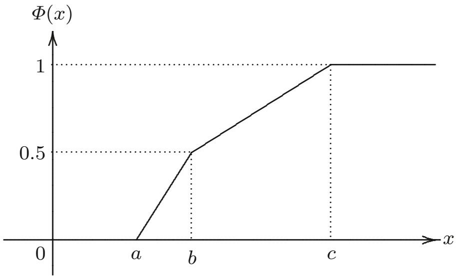

Fig. 1 Zigzag uncertainty distribution

Definition 3 (Liu 2009) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$M\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Definition 4 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = M\{\xi \le x\} \tag{2}$$

for any real number x (Fig. 1).

Definition 5 (Liu 2007) The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^{0} M\{\xi \le x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite.

Example 1 An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x-a)/2(b-a), & \text{if } a \le x < b \\ (x+c-2b)/2(c-b), & \text{if } b \le x < c \\ 1, & \text{if } x \ge c \end{cases} \tag{3}$$

denoted by $\mathcal{Z}(a, b, c)$ where a, b, c are real numbers with $a < b < c$.

Definition 6 (Liu 2010) An uncertainty distribution Φ of an uncertain variable ξ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$. In this case, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Example 2 The inverse uncertainty distribution of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha \ge 0.5. \end{cases}$$

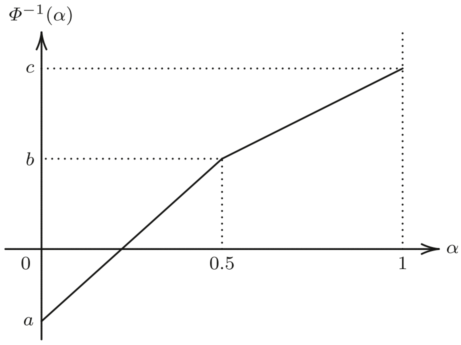

Fig. 2 Inverse zigzag uncertainty distribution

Theorem 1 (Liu 2010) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{4}$$

is an uncertain variable with inverse uncertainty distribution (Fig. 2)

$$\Psi^{-1} = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)). \tag{5}$$

Example 3 Let ξ be an uncertain variable with regular uncertainty distribution Φ. Since $f(x) = ax + b$ is a strictly increasing function for any constants $a > 0$ and b, the inverse uncertainty distribution of $a\xi + b$ is

$$\Psi^{-1}(\alpha) = a\Phi^{-1}(\alpha) + b. \tag{6}$$

Example 4 Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Since

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$

is a strictly increasing function, the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{7}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha). \tag{8}$$

Theorem 2 (Liu 2010) *Assume that the constraint function $g(x, \xi_1, \xi_2, \ldots, \xi_n)$ strictly increases with respect to $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreases with respect to $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$. If $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, then the chance constraint*

$$M\{g(x, \xi_1, \xi_2, \ldots, \xi_n) \le 0\} \ge \alpha \tag{9}$$

holds if and only if

$$g(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) \le 0. \tag{10}$$

3 Uncertain DEA model

This section will introduce an uncertain DEA model proposed by Wen et al. (2014). The symbols and notations are given as follows:

DMU_i:	the ith DMU, $i = 1, 2, \ldots, n$;
DMU_0:	the target DMU;
$\widetilde{x}_k = (\widetilde{x}_{k1}, \widetilde{x}_{k2}, \ldots, \widetilde{x}_{kp})$:	the uncertain inputs vector of DMU_k, $k = 1, 2, \ldots, n$;
$\Phi_{ki}(x)$:	the uncertainty distribution of $\widetilde{x}_{ki}$, $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, p$;
$\widetilde{x}_0 = (\widetilde{x}_{01}, \widetilde{x}_{02}, \ldots, \widetilde{x}_{0p})$:	the inputs vector of the target DMU_0;
$\Phi_{0i}(x)$:	the uncertainty distribution of $\widetilde{x}_{0i}$, $i = 1, 2, \ldots, p$;
$\widetilde{y}_k = (\widetilde{y}_{k1}, \widetilde{y}_{k2}, \ldots, \widetilde{y}_{kq})$:	the uncertain output vector of DMU_k, $k = 1, 2, \ldots, n$;
$\Psi_{kj}(x)$:	the uncertainty distribution of $\widetilde{y}_{kj}$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, q$;
$\widetilde{y}_0 = (\widetilde{y}_{01}, \widetilde{y}_{02}, \ldots, \widetilde{y}_{0q})$:	the outputs vector of the target DMU_0;
$\Psi_{0j}(x)$:	the uncertainty distribution of $\widetilde{y}_{0j}$, $j = 1, 2, \ldots, q$.

The model Wen et al. (2014) is given as:

$$\begin{cases} \max \quad \sum_{i=1}^{p} s_i^- + \sum_{j=1}^{q} s_j^+ \\ \text{subject to :} \\ \qquad M\left\{\sum_{k=1}^{n} \widetilde{x}_{ki}\lambda_k \le \widetilde{x}_{0i} - s_i^-\right\} \ge \alpha, \quad i = 1, 2, \ldots, p \\ \qquad M\left\{\sum_{k=1}^{n} \widetilde{y}_{kj}\lambda_k \ge \widetilde{y}_{0j} + s_j^+\right\} \ge \alpha, \quad j = 1, 2, \ldots, q \\ \qquad \sum_{k=1}^{n} \lambda_k = 1 \\ \qquad \lambda_k \ge 0, \quad k = 1, 2, \ldots, n \\ \qquad s_i^- \ge 0, \quad i = 1, 2, \ldots, p \\ \qquad s_j^+ \ge 0, \quad j = 1, 2, \ldots, q \end{cases} \tag{11}$$

in which M is the uncertainty measure introduced in Sect. 2. The objective of the model is to maximize the total slacks in inputs and outputs subject to some chance constraints.

Definition 7 (Wen et al. 2014, α-efficiency) DMU_0 is α-efficient if $s_i^{-*}(\alpha)$ and $s_j^{+*}(\alpha)$ are zero for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, where $s_i^{-*}(\alpha)$ and $s_j^{+*}(\alpha)$ are optimal solutions of (11).

This definition aligns more closely with additive-efficiency definition. However, it differs in that uncertain measure is involved. For instance, as determined by the choice of α, there is a risk that DMU_0 will not be efficient even when the condition of Definition 7 is satisfied.

4 Stable regions of inefficient DMUs

This section will give some theories, based on which the stable regions of the inefficient DMUs evaluated by an uncertain DEA model (11) can be obtained.

Theorem 3 *If DMU_0 is α-inefficient, then the optimal solution satisfying $\lambda_0^* = 0$.*

Proof We assume that the evaluated target is DMU_1. That is, we should prove $\lambda_1 = 0$. For a fixed α, suppose the optimal solution is $(\boldsymbol{\lambda}^*, s^{-*}, s^{+*})$ and the optimal objective value is $\sum_{i=1}^{p} s_i^{-*} + \sum_{j=1}^{q} s_j^{+*}$. If $\lambda_1^* = 0$, then the Theorem has been proved. Otherwise, let $\lambda_1 > 0$. Since DMU_1 is inefficient, there exists at least one $s_i^{-*} > 0$ or $s_j^{+*} > 0$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. Without loss of generality, we assume that $s_1^{-*} > 0$. If $\lambda_1^* = 1$, then $M\left\{\widetilde{x}_{11} \le \widetilde{x}_{11} - s_1^{-*}\right\} = 0$. The contradiction implies that $\lambda_1^* \ne 1$. That is, $0 < \lambda_1^* < 1$. Then it can be obtained that

$$\begin{aligned} & M\left\{\sum_{k=1}^{n} \widetilde{x}_{ki}\lambda_k^* \le \widetilde{x}_{1i} - s_i^{-*}\right\} \\ & = M\left\{\sum_{k=2}^{n} \widetilde{x}_{ki}\lambda_k^* \le (1-\lambda_1^*)\widetilde{x}_{1i} - s_i^{-*}\right\} \\ & = M\left\{\frac{\sum_{k=2}^{n} \widetilde{x}_{ki}\lambda_k^*}{1-\lambda_1^*} \le \widetilde{x}_{1i} - \frac{s_i^{-*}}{(1-\lambda_1^*)}\right\} \\ & \ge \alpha \end{aligned}$$

for all $i = 1, 2, \ldots, p$. Similarly, we can get

$$\begin{aligned} & M\left\{\sum_{k=1}^{n} \widetilde{y}_{kj}\lambda_k^* \ge \widetilde{y}_{1j} + s_j^{+*}\right\} \\ & = M\left\{\frac{\sum_{k=2}^{n} \widetilde{y}_{kj}\lambda_k^*}{1-\lambda_1^*} \ge \widetilde{y}_{1j} + \frac{s_j^{+*}}{(1-\lambda_1^*)}\right\} \\ & \ge \alpha \end{aligned}$$

for all $j = 1, 2, \ldots, q$. Since $\frac{\sum_{k=2}^{n} \lambda_k^*}{1-\lambda_1^*} = 1$, $\left(0, \frac{\lambda_2^*}{\sum_{k=2}^{n} \lambda_k^*}, \frac{\lambda_3^*}{\sum_{k=2}^{n} \lambda_k^*}, \ldots, \frac{\lambda_n^*}{\sum_{k=2}^{n} \lambda_k^*}\right)$ is a feasible solution. Then the objective value is $\frac{1}{1-\lambda_1^*}\Big(\sum_{i=1}^{p} s_i^{-*}$

$+\sum_{j=1}^{q} s_j^{+*}\Big) > \sum_{i=1}^{p} s_i^{-*} + \sum_{j=1}^{q} s_j^{+*}$, since $0 < \lambda_1^* < 1$, which leads to a contradiction with the assumption. Thus $\lambda_1^* = 0$. The theorem is proved.

Theorem 4 *If a DMU with $(\widetilde{x}_0, \widetilde{y}_0)$ is inefficient after evaluating by model (11), the new DMU with $(\widehat{x}_0, \widehat{y}_0) = (\widetilde{x}_0 - s^{-*}, \widetilde{y}_0 + s^{+*})$ is α-efficient, in which s^{-*} and s^{+*} are optimal solutions of (11).*

Proof The efficiency of $(\widehat{x}_0, \widehat{y}_0)$ is evaluated by solving the problem below:

$$\begin{cases} \max \ \sum_{i=1}^{p} s_i^- + \sum_{j=1}^{q} s_j^+ \\ \text{subject to :} \\ \qquad M\left\{\sum_{k=1,k\neq 0}^{n} \widetilde{x}_{ki}\lambda_k + \widehat{x}_{0i}\lambda_0 \le \widehat{x}_{0i} - s_i^-\right\} \ge \alpha, \quad i = 1,2\ldots,p \\ \qquad M\left\{\sum_{k=1,k\neq 0}^{n} \widetilde{y}_{kj}\lambda_k + \widehat{y}_{0j}\lambda_0 \ge \widehat{y}_{0j} + s_j^+\right\} \ge \alpha, \quad j = 1,2,\ldots,q \\ \qquad \sum_{k=1}^{n} \lambda_k = 1 \\ \qquad \lambda_k \ge 0, \quad k = 1,2,\ldots,n \\ \qquad s_i^- \ge 0, \quad i = 1,2,\ldots,p \\ \qquad s_j^+ \ge 0, \quad j = 1,2\ldots,q. \end{cases} \tag{12}$$

Let an optimal solution be $(\widehat{\lambda}, \widehat{s}^+, \widehat{s}^-)$. Suppose the DMU with $(\widehat{x}_0, \widehat{y}_0)$ is inefficient, then $\lambda_0 = 0$ can be got by Theorem 3. By inserting the formula $(\widehat{x}_0, \widehat{y}_0) = (\widetilde{x}_0 - s^{-*}, \widetilde{y}_0 + s^{+*})$ into constraints, we have

$$M\left\{\sum_{k=1,k\neq 0}^{n} \widetilde{x}_{ki}\widehat{\lambda}_k \le \widetilde{x}_{0i} - \widehat{s}_i^- - s_i^{-*}\right\} \ge \alpha,$$
$$i = 1,2\ldots,p,$$

$$M\left\{\sum_{k=1,k\neq 0}^{n} \widetilde{y}_{kj}\widehat{\lambda}_k \ge \widetilde{y}_{0j} + \widehat{s}_j^+ + s_j^{+*}\right\} \ge \alpha,$$
$$j = 1,2,\ldots,q.$$

Now, we can also write the constraints as

$$M\left\{\sum_{k=1,k\neq 0}^{n} \widetilde{x}_{ki}\widehat{\lambda}_k \le \widetilde{x}_{0i} - \widetilde{s}_i^-\right\} \ge \alpha, \quad i = 1,2\ldots,p,$$

$$M\left\{\sum_{k=1,k\neq 0}^{n} \widetilde{y}_{kj}\widehat{\lambda}_k \ge \widetilde{y}_{0j} + \widetilde{s}_j^+\right\} \ge \alpha, \quad j = 1,2,\ldots,q$$

where $\widetilde{s}^+ = \widehat{s}^+ + s^{+*} \ge 0$ and $\widetilde{s}^- = \widehat{s}^- + s^{-*} \ge 0$. Furthermore, we have

$$\sum_{i=1}^{p} \widetilde{s}_i^- + \sum_{j=1}^{q} \widetilde{s}_j^+ = \left(\sum_{i=1}^{p} \widehat{s}_i^- + s_i^{-*}\right) + \left(\sum_{j=1}^{q} \widehat{s}_j^+ + s_j^{+*}\right) \le \sum_{i=1}^{p} s_i^{-*} \sum_{j=1}^{q} s_j^{+*}$$

since these constraints are a feasible solution for the model (11) and $\sum_{i=1}^{p} s_i^{-*} + \sum_{j=1}^{q} s_j^{+*}$ is maximal. It follows that we have $\sum_{i=1}^{p} \widehat{s}_i^- + \sum_{j=1}^{q} \widehat{s}_j^+ = 0$ which implies that all components $\widehat{s}_i^-$ and $\widehat{s}_j^+$ are zero. Hence, fuzzy efficiency is achieved as claimed.

5 Stable regions of efficient DMUs

This section will give some stability analysis to the efficient DMUs evaluated by the uncertain DEA model (11). To get the efficient radius of the efficient DMUs, we will give the following model:

$$
\begin{cases}
\min \quad \sum_{i=1}^{p} t_i^+ + \sum_{j=1}^{q} t_j^- \\
\text{subject to :} \\
\qquad M\left\{ \sum_{k=1,k\neq 0}^{n} \tilde{x}_{ki}\lambda_k \le \tilde{x}_{0i} + t_i^+ \right\} \ge \alpha, \quad i = 1, 2, \ldots, p \\
\qquad M\left\{ \sum_{k=1,,k\neq 0}^{n} \tilde{y}_{kj}\lambda_k \ge \tilde{y}_{0j} - t_j^- \right\} \ge \alpha, \quad j = 1, 2, \ldots, q \\
\qquad \sum_{k=1}^{n} \lambda_k = 1 \\
\qquad \lambda_k \ge 0, \quad k = 1, 2, \ldots, n \\
\qquad t_i^+ \ge 0, \quad i = 1, 2, \ldots, p \\
\qquad t_j^- \ge 0, \quad j = 1, 2, \ldots, q.
\end{cases}
\tag{13}
$$

Theorem 5 *For a fixed α, the efficient DMU_0 stays efficient if $(\widehat{x}_0, \widehat{y}_0) = (\tilde{x}_0 + t^{+*}, \tilde{y}_0 - t^{-*})$, where t^{+*} and t^{-*} are optimal solutions of* (13).

Proof Consider the following DEA model for evaluating the relative efficiency of the adjusted DMU_0:

Let the optimal solution be $(\lambda_j^*, \lambda_0^*, s^{-*}, s^{+*})$ and assume that the DMU_0 is inefficient. From Theorem 3, we get $\lambda_0^* = 0$. Then model (14) has the following form:

$$
\begin{cases}
\max \quad \sum_{i=1}^{p} s_i^- + \sum_{j=1}^{q} s_j^+ \\
\text{subject to :} \\
\qquad M\left\{ \sum_{k=1,k\neq 0}^{n} \tilde{x}_{ki}\lambda_k + (\tilde{x}_{0i} + t_i^{+*})\lambda_0 \le (\tilde{x}_{0i} + t_i^{+*}) - s_i^- \right\} \ge \alpha, \quad i = 1, 2, \ldots, p \\
\qquad M\left\{ \sum_{k=1,k\neq 0}^{n} \tilde{y}_{kj}\lambda_k + (\tilde{y}_{0j} - t_j^{-*})\lambda_0 \ge (\tilde{y}_{0j} - t_j^{-*}) + s_j^+ \right\} \ge \alpha, \quad j = 1, 2, \ldots, q \\
\qquad \sum_{k=1}^{n} \lambda_k = 1 \\
\qquad \lambda_k \ge 0, \quad k = 1, 2, \ldots, n \\
\qquad s_i^- \ge 0, \quad i = 1, 2, \ldots, p \\
\qquad s_j^+ \ge 0, \quad j = 1, 2, \ldots, q.
\end{cases}
\tag{14}
$$

$$
\begin{cases}
\max \quad \sum_{i=1}^{p} s_i^- + \sum_{j=1}^{q} s_j^+ \\
\text{subject to :} \\
\qquad M\left\{ \sum_{k=1,k\neq 0}^{n} \tilde{x}_{ki}\lambda_k \le \tilde{x}_{0i} + (t_i^{+*} - s_i^-) \right\} \ge \alpha, \quad i = 1, 2, \ldots, p \\
\qquad M\left\{ \sum_{k=1,k\neq 0}^{n} \tilde{y}_{kj}\lambda_k \ge \tilde{y}_{0j} - (t_j^{-*} - s_j^+) \right\} \ge \alpha, \quad j = 1, 2, \ldots, q \\
\qquad \sum_{k=1}^{n} \lambda_k = 1 \\
\qquad \lambda_k \ge 0, \quad k = 1, 2, \ldots, n \\
\qquad s_i^- \ge 0, \quad i = 1, 2, \ldots, p \\
\qquad s_j^+ \ge 0, \quad j = 1, 2, \ldots, q
\end{cases}
\tag{15}
$$

whose optimal solution is a feasible solution of model (13). Hence, $t_i^{+*} - s_i^{-*} \geq t_i^{+*}$ and $t_j^{-*} - s_j^{+*} \geq t_j^{-*}$, which means that $s_i^{-*} = 0$ and s_j^{+*}, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. This leads to a contradiction with the assumption.

6 Computation of stable regions

The above two sections have given some theory analysis to stable regions, which is too complex to use conveniently in practice. Thus, this section will give some equivalent deterministic models to simplify the computation process.

In Wen et al. (2014), the uncertain model (11) has been proved to be equal to the following model:

$$\begin{cases} \max & \sum_{i=1}^{p} s_i^- + \sum_{j=1}^{q} s_j^+ \\ \text{subject to}: & \\ & \sum_{k=1,k\neq 0}^{n} \lambda_k \Phi_{ki}^{-1}(\alpha) + \lambda_0 \Phi_{0i}^{-1}(1-\alpha) \leq \Phi_{0i}^{-1}(1-\alpha) - s_i^-, \quad i = 1, 2, \ldots, p \\ & \sum_{k=1,k\neq 0}^{n} \lambda_k \Psi_{kj}^{-1}(1-\alpha) + \lambda_0 \Psi_{0j}^{-1}(\alpha) \geq \Psi_{0j}^{-1}(\alpha) + s_j^+, \quad j = 1, 2, \ldots, q \\ & \sum_{k=1}^{n} \lambda_k = 1 \\ & \lambda_k \geq 0, \quad k = 1, 2, \ldots, n \\ & s_i^- \geq 0, \quad i = 1, 2 \ldots, p \\ & s_j^+ \geq 0, \quad j = 1, 2, \ldots, q. \end{cases} \tag{16}$$

Similarly, it can be proved that the uncertain model (13) is equivalent to the following deterministic model:

$$\begin{cases} \min & \sum_{i=1}^{p} t_i^+ + \sum_{j=1}^{q} t_j^- \\ \text{subject to}: & \\ & \sum_{k=1,k\neq 0}^{n} \lambda_k \Phi_{ki}^{-1}(\alpha) \leq \Phi_{0i}^{-1}(1-\alpha) + t_i^+, \quad i = 1, 2, \ldots, p \\ & \sum_{k=1,k\neq 0}^{n} \lambda_k \Psi_{kj}^{-1}(1-\alpha) \geq \Psi_{0j}^{-1}(\alpha) - t_j^-, \quad j = 1, 2, \ldots, q \\ & \sum_{k=1}^{n} \lambda_k = 1 \\ & \lambda_k \geq 0, \quad k = 1, 2, \ldots, n \\ & t_i^+ \geq 0, \quad i = 1, 2, \ldots, p \\ & t_j^- \geq 0, \quad j = 1, 2, \ldots, q. \end{cases} \tag{17}$$

From the above analysis, the ranges of inputs and outputs and radius of stability of DMU_0 are identified as follows:

1. If DMU_0 is inefficient by solving model (16), then the inefficient region is $(\widetilde{x}_0 - s^{-*}, \widetilde{y}_0 + s^{+*})$, where s^{-*} and s^{+*} are optimal solutions of (16).
2. If DMU_0 is efficient by solving model (16), then we use model (17) to account for the efficient radius. The efficient region is $(\widetilde{x}_0 + t^{+*}, \widetilde{y}_0 - t^{-*})$, where t^{+*} and t^{-*} are optimal solutions of (17).

7 A numerical example

This numerical example is presented to give an illustration of the sensitivity and stability analysis. Table 1 provides the information of the DMUs. There are two uncertain inputs and two uncertain outputs which are all zigzag uncertain variables denoted by $\mathcal{Z}(a, b, c)$. In this example, we assume that $\alpha \geq 0.5$. Then we can get that the inverse uncertainty distribution of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is $\Phi^{-1}(\alpha) = (2 - 2\alpha)b + (2\alpha - 1)c$.

As an example, assume the evaluating target to be DMU_1. Then the uncertain DEA model (11) has the following form:

Table 1 DMUs with two zigzag uncertain inputs and two zigzag uncertain outputs

DMU_i	1	2	3	4	5
Input 1	$\mathcal{Z}(3.5, 4.0, 4.5)$	$\mathcal{Z}(2.9, 2.9, 2.9)$	$\mathcal{Z}(4.4, 4.9, 5.4)$	$\mathcal{Z}(3.4, 4.1, 4.8)$	$\mathcal{Z}(5.9, 6.5, 7.1)$
Input 2	$\mathcal{Z}(2.9, 3.1, 3.3)$	$\mathcal{Z}(1.4, 1.5, 1.6)$	$\mathcal{Z}(3.2, 3.6, 4.0)$	$\mathcal{Z}(2.1, 2.3, 2.5)$	$\mathcal{Z}(3.6, 4.1, 4.6)$
Output 1	$\mathcal{Z}(2.4, 2.6, 2.8)$	$\mathcal{Z}(2.2, 2.2, 2.2)$	$\mathcal{Z}(2.7, 3.2, 3.7)$	$\mathcal{Z}(2.5, 2.9, 3.3)$	$\mathcal{Z}(4.4, 5.1, 5.8)$
Output 2	$\mathcal{Z}(3.8, 4.1, 4.4)$	$\mathcal{Z}(3.3, 3.5, 3.7)$	$\mathcal{Z}(4.3, 5.1, 5.9)$	$\mathcal{Z}(5.5, 5.7, 5.9)$	$\mathcal{Z}(6.5, 7.4, 8.3)$

Table 2 Evaluating the results with $\alpha = 0.6$

DMUs	$(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*, \lambda_5^*)$	$(s_1^{-*}, s_2^{-*}, s_1^{+*}, s_2^{+*})$	The evaluating result
DMU_1	(0, 0.25, 0, 0.75, 0)	(0, 0.93, 0.02, 0.94)	Inefficiency
DMU_2	(0, 1, 0, 0, 0)	(0, 0, 0, 0)	Efficiency
DMU_3	(0, 0, 0, 0.78, 0.22)	(0.03, 0.76, 0, 0.75)	Inefficiency
DMU_4	(0, 0, 0, 1, 0)	(0, 0, 0, 0)	Efficiency
DMU_5	(0, 0, 0, 0, 1)	(0, 0, 0, 0)	Efficiency

$$\begin{cases} \max \quad s_1^- + s_2^- + s_1^+ + s_2^+ \\ \text{subject to :} \\ \lambda_1[(1-2\alpha)\times 3.5 + (1-2\alpha)\times 4] + \lambda_2[(2-2\alpha)\times 2.9 + (2\alpha-1)\times 2.9] \\ \quad +\lambda_3[(2-2\alpha)\times 4.9 + (2\alpha-1)\times 5.4] + \lambda_4[(2-2\alpha)\times 4.1 + (2\alpha-1)\times 4.8] \\ \quad +\lambda_5[(2-2\alpha)\times 6.5 + (2\alpha-1)\times 7.1] \le (1-2\alpha)\times 3.5 + (1-2\alpha)\times 4 - s_1^- \\ \lambda_1[(1-2\alpha)\times 2.9 + (1-2\alpha)\times 3.1] + \lambda_2[(2-2\alpha)\times 1.5 + (2\alpha-1)\times 1.6] \\ \quad +\lambda_3[(2-2\alpha)\times 3.6 + (2\alpha-1)\times 4] + \lambda_4[(2-2\alpha)\times 2.3 + (2\alpha-1)\times 2.5] \\ \quad +\lambda_5[(2-2\alpha)\times 4.1 + (2\alpha-1)\times 4.6] \le (1-2\alpha)\times 2.9 + (1-2\alpha)\times 3.1 - s_2^- \\ \lambda_1[(2-2\alpha)\times 2.6 + (2\alpha-1)\times 2.8] + \lambda_2[(1-2\alpha)\times 2.2 + (1-2\alpha)\times 2.2] \\ \quad +\lambda_3[(1-2\alpha)\times 2.7 + (1-2\alpha)\times 3.2] + \lambda_4[(1-2\alpha)\times 2.5 + (1-2\alpha)\times 2.9] \\ \quad +\lambda_5[(1-2\alpha)\times 4.4 + (1-2\alpha)\times 5.1] \ge (2-2\alpha)\times 2.6 + (2\alpha-1)\times 2.8 + s_1^+ \\ \lambda_1[(2-2\alpha)\times 4.1 + (2\alpha-1)\times 4.4] + \lambda_2[(1-2\alpha)\times 3.3 + (1-2\alpha)\times 3.5] \\ \quad +\lambda_3[(1-2\alpha)\times 4.3 + (1-2\alpha)\times 5.1] + \lambda_4[(1-2\alpha)\times 5.5 + (1-2\alpha)\times 5.7] \\ \quad +\lambda_5[(1-2\alpha)\times 6.5 + (1-2\alpha)\times 7.4] \ge (2-2\alpha)\times 4.1 + (2\alpha-1)\times 4.4 + s_2^+ \\ \sum_{k=1}^{5} \lambda_k = 1 \\ \lambda_k \ge 0, \quad k = 1, 2, \ldots, 5 \\ s_1^- \ge 0, \quad s_2^- \ge 0, \\ s_1^+ \ge 0, \quad s_2^+ \ge 0. \end{cases} \tag{18}$$

When we set $\alpha = 0.6$, the results evaluated by model (18) are shown in Table 2. The results can be interpreted in the following way: DMU_1 and DMU_3 are inefficient, whereas DMU_2, DMU_4 and DMU_5 are efficient.

Stable regions of inefficient DMUs For DMU_1 and DMU_3, we will give their stable regions according to Theory 4. Table 3 reports the sensitivity analysis results for DMU_1 and DMU_3. In Table 3, the columns 2 and 3 report lower bounds of variation ranges of inputs and the columns 4 and 5 are upper bounds of variation ranges of outputs. For instance, DMU_1 stays inefficient when $(\widehat{x}_{11}, \widehat{x}_{12}, \widehat{y}_{11}, \widehat{y}_{12}) = (\widetilde{x}_{11}, \widetilde{x}_{12} - r_{x2}, \widetilde{y}_{11} + r_{y1}, \widetilde{y}_{12} + r_{y2})$, in which $0 \le r_{x2} < 0.93$, $0 \le r_{y1} < 0.02$ and $0 \le r_{y2} < 0.94$.

Table 4 shows the lower bounds of input 1 and input 2 for DMU_1 and DMU_3, and the upper bounds of outputs are shown in Table 5. For instance, input 2 in DMU_1 reduces to $\mathcal{Z}(2.1, 2.3, 2.5)$ from $\mathcal{Z}(2.9, 3.1, 3.3)$, and other input and outputs remain the same. Since $\mathcal{Z}(2.1, 2.3, 2.5)$ has not exceeded the lower bound $\mathcal{Z}(1.97, 2.17, 3.37)$, it remains

Table 3 Sensitivity analysis results for DMU_1 and DMU_3

DMUs	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}
DMU_1	0	0.93	0.02	0.94
DMU_3	0.03	0.76	0.00	0.75

Table 4 Lower bounds of inputs for DMU_1 and DMU_3

DMUs	Input 1	Input 2
DMU_1	$\mathcal{Z}(3.5, 4.0, 4.5)$	$\mathcal{Z}(1.97, 2.17, 3.37)$
DMU_3	$\mathcal{Z}(4.37, 4.87, 5.37)$	$\mathcal{Z}(2.44, 2.84, 3.24)$

Table 5 Upper bounds of outputs for DMU_1 and DMU_3

DMUs	Output 1	Output 2
DMU_1	$\mathcal{Z}(2.42, 2.62, 2.82)$	$\mathcal{Z}(4.74, 5.04, 5.34)$
DMU_3	$\mathcal{Z}(2.7, 3.2, 3.7)$	$\mathcal{Z}(5.05, 5.85, 6.65)$

Table 6 Sensitivity analysis results for DMU_2, DMU_4 and DMU_5

DMUs	t_1^{+*}	t_2^{+*}	t_1^{-*}	t_2^{-*}
DMU_2	0	0.23	0.16	0.00
DMU_4	0	0.02	0.00	1.21
DMU_5	0.00	0.00	3.11	2.06

Table 7 Upper bounds of inputs for DMU_2, DMU_4 and DMU_5

DMUs	Input 1	Input 2
DMU_2	$\mathcal{Z}(2.9, 2.9, 2.9)$	$\mathcal{Z}(1.63, 1.73, 1.83)$
DMU_4	$\mathcal{Z}(3.4, 4.1, 4.8)$	$\mathcal{Z}(2.12, 2.32, 2.52)$
DMU_5	$\mathcal{Z}(5.9, 6.5, 7.1)$	$\mathcal{Z}(3.6, 4.1, 4.6)$

inefficient. Let us consider DMU_3 when output 2 increases from $\mathcal{Z}(4.3, 5.1, 5.9)$ to $\mathcal{Z}(5.3, 6.1, 6.9)$. It becomes efficient because the new output $\mathcal{Z}(5.3, 6.1, 6.9)$ has exceeded the upper bound $\mathcal{Z}(2.44, 2.84, 3.24)$.

Stable regions of efficient DMUs For DMU_2, DMU_4 and DMU_5, we will use model (17) to get their stable regions. Similar to model (18), the model (17) can be converted to a similar form which is omitted here. Then we can get the stable regions shown in Table 6. Columns 2 and 3 report upper bounds of variation ranges of inputs and columns 4 and 5 are lower bounds of variation ranges of outputs. For instance, DMU_4 in Table 4 stays efficient when $(\widehat{x}_{41}, \widehat{x}_{42}, \widehat{y}_{41}, \widehat{y}_{42}) = (\widetilde{x}_{41}, \widetilde{x}_{42} + r_{x2}, \widetilde{y}_{41}, \widetilde{y}_{42} - r_{y2})$, in which $0 \le r_{x2} \le 0.02$ and $0 \le r_{y2} \le 1.21$.

Table 7 shows the upper bounds of inputs for DMU_2, DMU_4 and DMU_5, and the lower bounds of outputs are shown in Table 8. For instance, input 2 in DMU_2 increases to $\mathcal{Z}(1.82, 1.92, 2.02)$ from $\mathcal{Z}(1.4, 1.5, 1.6)$, and other input and outputs remain the same. Since $\mathcal{Z}(1.82, 1.92, 2.02)$ has exceeded the upper bound $\mathcal{Z}(1.63, 1.73, 1.83)$, it becomes inefficient. Let us consider DMU_5 when output 2 decreases from $\mathcal{Z}(6.5, 7.4, 8.3)$ to $\mathcal{Z}(4.76, 5.66, 6.56)$. It remains efficient because the new output $\mathcal{Z}(4.76, 5.66, 6.56)$ has not exceeded the lower bound $\mathcal{Z}(4.44, 5.34, 6.24)$.

Table 8 Lower bounds of outputs for DMU_2, DMU_4 and DMU_5

DMUs	Output 1	Output 2
DMU_2	$\mathcal{Z}(2.04, 2.04, 2.04)$	$\mathcal{Z}(3.3, 3.5, 3.7)$
DMU_4	$\mathcal{Z}(2.5, 2.9, 3.3)$	$\mathcal{Z}(4.29, 4.49, 4.69)$
DMU_5	$\mathcal{Z}(1.29, 1.99, 2.69)$	$\mathcal{Z}(4.44, 5.34, 6.24)$

8 Conclusion

This paper mainly gave a sensitivity and stability analysis of the additive model for the data envelopment analysis with uncertain inputs and outputs. Some theories based on uncertain additive model, as well as the stable regions of all the DMUs were obtained. To use it conveniently in practice, some computation methods of the stable regions were given. Finally, the sensitivity and stability of the uncertain additive model were illustrated through a numerical example.

Several further studies should be considered. For one thing, as introduced in the introduction section, we will try to apply the cognitive approach to DEA. The uncertain input and output data can be processed by cognitive analysis in advance, and then the new deterministic data may be evaluated by the DEA method. In addition, we will apply the uncertain DEA to evaluate the material support plans in support systems.

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Joint optimization of LORA and spares stocks considering corrective maintenance time

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Abstract: Level of repair analysis (LORA) is an important method of maintenance decision for establishing systems of operation and maintenance in the equipment development period. Currently, the research on equipment of repair level focuses on economic analysis models which are used to optimize costs and rarely considers the maintenance time required by the implementation of the maintenance program. In fact, as to the system requiring high mission complete success, the maintenance time is an important factor which has a great influence on the availability of equipment systems. Considering the relationship between the maintenance time and the spares stocks level, it is obvious that there are contradictions between the maintenance time and the cost. In order to balance these two factors, it is necessary to build an optimization LORA model. To this end, the maintenance time representing performance characteristic is introduced, and on the basis of spares stocks which is traditionally regarded as a decision variable, a decision variable of repair level is added, and a multi-echelon multi-indenture (MEMI) optimization LORA model is built which takes the best cost-effectiveness ratio as the criterion, the expected number of backorder (EBO) as the objective function and the cost as the constraint. Besides, the paper designs a convex programming algorithm of multi-variable for the optimization model, provides solutions to the non-convex objective function and methods for improving the efficiency of the algorithm. The method provided in this paper is proved to be credible and effective according to the numerical example and the simulation result.

Keywords: level of repair, spares, convex optimization, multi-echelon multi-indenture (MEMI).

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1. Introduction

As we all know, equipment such as aircraft is expensive and technically complex, with a high downtime cost. Before the equipment is deployed, several tactical level decisions concerning its corrective maintenance need to be made: (i) which components to repair upon failure and which to discard, (ii) where to perform the repair, (iii) the amount of spares to stock at each site in the repair network. These decisions should be solved so that a target availability of the installed base is achieved against the lowest cost. Generally, the first two decisions are taken explicitly through level of repair analysis (LORA) and the third decision can be solved by spares stocks through general multi-echelon technique for recoverable item control (METRIC) type methods.

LORA is to decide the repair level of the product of downtime as soon as possible caused by maintenance with certain cost constraints in order to improve system availability and reduce maintenance cost as much as possible. Generally, LORA is carried out in the sequence of non-economic analysis and then economic analysis to identify all the objects' repair level. There are many qualitative constraints in the non-economic analysis, such as motility requirements of deployment, restricts of current support systems, security requirements, special transportation requirements, practicability of repair technique, confidentiality constraints, and personnel and technical levels. However, the repair level of some products cannot be identified only by these qualitative constraint conditions in the product detailed design phase, that is to say, there can be either lack of these products' constraints or constraints existing in several repair levels. For reasons outlined above, additional quantitative methods are needed, like the economic analysis method proposed early [1,2], which decide the repair level of products mainly by comparing maintenance costs of spares, personnel, materials, support equipment and facilities, and training in different repair levels. LORA needs to consider many influence factors, but a majority of the influence factors are used in the non-economic analysis method, so it is not necessary to consider all factors in the optimization modeling. At the beginning of the product design phase, too much data really cannot be obtained. However, the engineering background of this paper is in the product detailed design phase, and the input

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data involved in the proposed model, including component failure rate, repair time, transport time, spare unit cost and maintenance resource cost except spares, can totally be obtained by design and analysis methods of reliability and maintainability in the detailed design phase. For example, the component failure rate can be output from reliability prediction and allocation, the repair time can be output from maintainability prediction and allocation, the spare unit cost and maintenance resource cost except spares can be acquired from support resources planning and cost analysis, and the transport time can be output from early fielding analysis. This paper aims at the products whose repair level cannot be decided by qualitative analysis and takes the quantitative analysis and optimization to obtain the optimal repair level.

Generally, the quantitative analysis, namely economic analysis mentioned above, is only aimed at repair levels without considering several stations in the same level. In the support organization of multi-echelon multi-station, spare stock analysis is also to reduce maintenance downtime when failures occur and a certain amount of spare stock allocation is needed in each station. However, according to the overall target, there is a restrictive relation between the repair level and the spare stock. On the premise of the same amount of spares in different repair levels, repairing failure parts in lower stations can cause shorter maintenance downtime but higher maintenance cost, since for maintenance of a certain product, the amount of resources needed to be allocated in upper stations is much smaller than the total of sub-stations under the condition that an upper station supports many substations. However, under certain cost constraints, the spare allocation will be reduced thus the maintenance waiting time will be increased. It shows that respective optimizations for the repair level and spares will reduce the optimal effectiveness respectively. Only by establishing a joint optimal model with considerations of transferring and coupling relations of spare demands in the support organization of multi-echelon multi-station to synthetically balance the restrictive degree of the repair level and the spare allocation, can the global optimal maintenance plan be realized.

Current researches on optimal models of LORA are focused on repair levels of products that cannot be identified by qualitative factors, and take the maintenance cost as the optimal object to identify the repair levels of these products. And prior researches mainly optimize the repair level and stock allocation respectively without considering the interaction between these two optimal issues.

Barros [3] proposed the first multi-echelon multi-indenture (MEMI) LORA model. Barros assumes that the same decisions are taken at all locations at one echelon level and all components at a certain indenture level require the same resource and that the resources are infinite and she models the problem as an integer linear programming model which she solves by using Cplex. Barros and Riley [4] used the same model as Barros did and solved it by using a branch-and-branch approach. Saranga and Dinesh Kumar [5] made the same assumptions as Barros [3], except that each component requires its own unique resources. They solved the model by using a genetic algorithm. Basten et al. [6] generalized the two aforementioned models by allowing for components requiring multiple resources and multiple components requiring the same resource. Finally, Basten et al. [7] generalized the model of [6] by allowing for different decisions at various locations at one echelon level. They show that the LORA problem can be modeled efficiently as a generalized minimum cost flow model. Basten et al. [8] proposed a number of extensions to the model of [7]. Brick and Uchoa [9] also used similar assumptions as those in [7], but considered resources have a maximum capacity and assumed two indenture levels.

In the literature about spares, the METRIC type models are the most clastic. Sherbrook [10] developed the METRIC model, which is the basis for a huge stream of METRIC type models. The goal in these models is to find the most cost effective allocation of spares in a network. Muckstadt [11] extended the work of Sherbrooke (two-echelon, single-indenture) by allowing for two indenture levels, leading to the so-called MOD-METRIC model. Simon [12] considered the two-echelon, single-indenture, single-item problem, which is later extended to the general multi-echelon problem by Kruse [13]. Graves [14] proposed a more accurate approximation for the two-echelon, single-indenture problem, the VARI-METRIC model, which Sherbrooke [15] extended to the two-indenture level. Axsater [16] provided an exact evaluation and enumeration but with penalty costs instead of a service level constraint. Rustenburg et al. [17] gave an exact evaluation for the general MEMI problem. Kim [18] proposed an algorithm for a multi-echelon repairable item stocks system with depot spares and general repair time distribution.

Some researchers solve the problem of LORA and spares stocks jointly. Alfredsson [19] firstly proposed a two-echelon, single-indenture model joint LORA and spares stocks. He assumes it is required at the same location that each component requires one resource and all components require the same resource. Basten et al. [20] proposed the same model as that in [19], but they allowed for more general component-resource and components may share resources.

However, in the existing literature about spares inventory and LORA, a majority of models mainly consider how to minimize the costs, while disregard the maintenance time. Although Alfredsson [19] considered the system downtime, he omitted the spares waiting time which is the major part of the downtime. In this paper, we take a research on the joint optimization of LORA and spares stocks by considering the maintenance time resulting from maintenance action. In the spares inventory, higher maintenance time will lead a larger number of spares to be stocked to achieve the same availability. Availability is defined by the up time (mean time between failure, MTBF) and the down time (mean time due to spares, maintenance and other delays resulting from maintenance action, MDT) [21]. As is known, availability is an important factor in equipment systems, which is significantly influenced by the maintenance time. Since MTBF is a reliability parameter related to the equipment's own design away from the influence of the optimization, MDT is the main factor to affect the availability of the equipment. Therefore, MDT can represent the system effectiveness instead of the availability of the equipment in our optimization model. Disregarding human factors or management delay in maintenance, MDT mainly means the turn-around time (TAT) which includes pure repair time, transportation time and spares waiting time. The pure repair time is the execution time depending on the specific steps to finish the job when all resources are ready, and it is generally a constant value plus the transportation time. Shortening the pure repair time or other resources waiting time generally has no obvious effect on the maintenance time, therefore, in the optimization, by shortening the spares waiting time, we can achieve a target availability of equipment systems, in other words, the optimization goal is to shorten the spares waiting time. According to the Little Theory, we can transform the spares waiting time into the expected number of backorder (EBO), and further we can transform the maintenance time-cost balance into the EBO-cost balance. In the model, we will take EBO as the objective function, maintenance cost as the constraint condition.

In Section 2, we outline our model and design a convex programming algorithm to solve the model in Section 3. In Section 4, we design a numerical experiment and confirm the correctness of our algorithm by simulation. Finally, we give some conclusions and recommendations for further research in Section 5.

2. Model

2.1 Assumptions and notations

We use the underlying assumptions in the MEMI LORA model:

(i) Line replaceable unit (LRU) failure time is exponential distribution;

(ii) For each component at each location, the $(s-1, s)$ continuous review stocks control policy is used;

(iii) Except for spares, other maintenance resources are always adequate;

(iv) A failure of LRU is caused by a failure in at most one shop replaceable unit (SRU).

We index the depot-level sites by Dc $(c = 1, 2, \dots, C)$. Each depot-level site supports multiple intermediate-level sites which are indexed by Ib $(b = 1, 2, \dots, B)$. Each intermediate-level site supports multiple organization-level sites which are indexed by Oa $(a = 1, 2, \dots, A)$. Let 0 denote the LRU and index the LRU by i $(i = 1, 2, \dots, I)$. Each LRU contains multiple SRUs which are indexed by x $(x = 1, 2, \dots, X)$.

m^{Oa}: the demand rate of the site Oa;

m^{L}_{i0}: the demand rate of the LRU_i at the site $L(L\colon Oa, Ib, Dc)$;

m^{L}_{ix}: the demand rate of the SRU_{ix} at the site $L(L\colon Oa, Ib, Dc)$;

p^{Oa}_{i1}: probability of the failure LRU_i at the site Oa repaired at organization-level;

p^{Oa}_{i2}: probability of the failure LRU_i at the site Oa delivered to intermediate-level to repair;

p^{Oa}_{i3}: probability of the failure LRU_i at the site Oa delivered to depot-level to repair;

p^{Ib}_{i2}: probability of the failure LRU_i at the site Ib repaired at intermediate-level;

p^{Ib}_{i3}: probability of the failure LRU_i at the site Ib delivered to depot-level to repair;

f^{Oa}_{ix1}: probability of the failure SRU_{ix} at the site Oa repaired at organization-level;

f^{Oa}_{ix2}: probability of the failure SRU_{ix} at the site Oa delivered to intermediate-level to repair;

f^{Oa}_{ix3}: probability of the failure SRU_{ix} at the site Oa delivered to depot-level to repair;

f^{Ib}_{ix2}: probability of the failure SRU_{ix} at the site Ib repaired at intermediate-level;

f^{Ib}_{ix3}: probability of the failure SRU_{ix} at the site Ib delivered to depot-level to repair;

q^{L}_{ix}: probability of the failure LRU_i at the site L caused by $\mathrm{SRU}_{ix}(L\colon Oa, Ib, Dc)$;

T^{L}_{i0}: repair time of the LRU_i at the site $L(L\colon Oa, Ib, Dc)$;

T^{L}_{ix}: repair time of the SRU_{ix} at the site $L(L\colon Oa, Ib, Dc)$;

T_w: spares waiting time of organization-level;

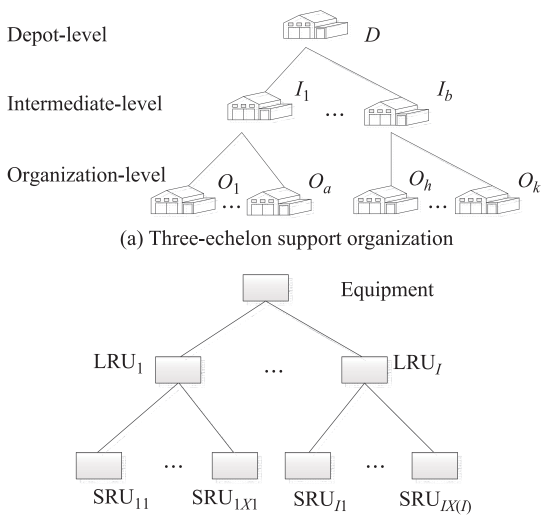
T^{L}_{wi0}: spares waiting time of the LRU_i at the site $L(L\colon Oa, Ib, Dc)$;

T^{L}_{wix}: spares waiting time of the SRU_{ix} at the site $L(L\colon Oa, Ib, Dc)$;

T_{i0}^{DcIb}: transportation time of the site Dc to the site Ib;
T_{i0}^{DcOa}: transportation time of the site Dc to the site Oa;
T_{i0}^{IbOa}: transportation time of the site Ib to the site Oa;
X_{i0}^{L}: the number of the LRU_i in repairing at the site $L(L\colon Oa, Ib, Dc)$;
X_{ix}^{L}: the number of the SRU_{ix} in repairing at the site $L(L\colon Oa, Ib, Dc)$;
s_{i0}^{L}: the stock level of the LRU_i at the site $L(L\colon Oa, Ib, Dc)$;
s_{i0}^{L}: the stock level of the SRU_{ix} at the site $L(L\colon Oa, Ib, Dc)$;
$\text{E}[X_{i0}^{L}]$: the expected pipeline for the LRU_i at the site $L(L\colon Oa, Ib, Dc)$;
$\text{E}[X_{ix}^{L}]$: the expected pipeline for the SRU_{ix} at the site $L(L\colon Oa, Ib, Dc)$;
EBO_i^{Oa}: the expected LRU_i backorder at the site Oa;
$\text{EBO}(s_{i0}^{L}|\text{E}[X_{i0}^{L}])$: the expected LRU_i backorder at the site L when the stock is s_{i0}^{L} and the expected pipeline is $\text{E}[X_{i0}^{L}](L\colon Oa, Ib, Dc)$;
$\text{EBO}(s_{ix}^{L}|\text{E}[X_{ix}^{L}])$: the expected SRU_{ix} backorder at the site L when the stock is s_{ix}^{L} and the expected pipeline is $\text{E}[X_{ix}^{L}](L\colon Oa, Ib, Dc)$.

2.2 Mathematical model

In this section, we mainly consider the three-echelon, two-indenture support system, and based on the analysis, we build a MEMI LORA optimization model as shown in Fig.1.



(a) Three-echelon support organization

(b) Two-indenture product structure

Fig. 1 Three-echelon, two-indenture support structure

As shown in Fig. 1(a), the support organization has three echelons, in which one depot-level site supports a number of repair sites called intermediate-level and one intermediate-level supports a number of repair sites called organization-level. The product structure can be abstracted as a two-indenture system, which is composed of multiple LRUs consisting of a number of SRUs in series, as shown in Fig. 1(b).

In Section 1, we have analyzed that the objective function of model is the EBO which is transformed by the spares waiting time. Therefore, the real objective function is the spares waiting time. Now, we deduce the relationship between the spares waiting time and the EBO.

In the multi-indenture support organization, as we know, the mean waiting time is expressed as

$$T_w = \sum_{a=1}^{A} m^{Oa} T_w^{Oa} / \sum_{a=1}^{A} m^{Oa}. \tag{1}$$

And T_w^{Oa} can be expressed as

$$T_w^{Oa} = \sum_{i=1}^{I(a)} m_{i0}^{Oa} T_{wi0}^{Oa} / \sum_{i=1}^{I(a)} m_{i0}^{Oa} \tag{2}$$

where $I(a)$ is the number of LRU_i at the site Oa. We can get T_{wi0}^{Oa} from the Little formula:

$$T_{wi0}^{Oa} = \text{EBO}_i^{Oa} / m_{i0}^{Oa}. \tag{3}$$

Combining (1), (2) and (3), we can get

$$T_w = \sum_{a=1}^{A} \sum_{i=1}^{I(a)} \text{EBO}_i^{Oa} / \sum_{a=1}^{A} \sum_{i=1}^{I(a)} m_{i0}^{Oa}. \tag{4}$$

As shown in (4), when the demand rate of each site is known, achieving the minimum possible mean waiting time is equal to achieving minimum possible $\sum_{a=1}^{A} \sum_{i=1}^{I(a)} \text{EBO}_i^{Oa}$, therefore, the objective function is $\sum_{a=1}^{A} \sum_{i=1}^{I(a)} \text{EBO}_i^{Oa}$. We analyze the objective function as shown in Fig. 2.

In Fig. 2, we present how to compute the objective function EBO. According to the Palm's Theorem [21,22], we can obtain the EBO formula.

$$\text{EBO}(s^L|\text{E}[X^L]) = \sum_{x=s^L+1}^{\infty} (x - s^L) \frac{(\text{E}[X^L])^x}{x!} \text{e}^{-\text{E}[X^L]} \tag{11}$$

Meantime, the demand rate of spares at each site can be computed as follows:

$$m_{i0}^{Oa} = m_{i00}^{Oa} p_{i1}^{Oa} \tag{12}$$

$$m_{ix}^{Oa} = m_{i0}^{Oa} q_{ix}^{Oa} f_{ix1}^{Oa} \tag{13}$$

$$m_{i0}^{Ib} = \sum_{Oa \subset Ib} m_{i00}^{Oa} p_{i2}^{Oa} p_{i2}^{Ib} \tag{14}$$

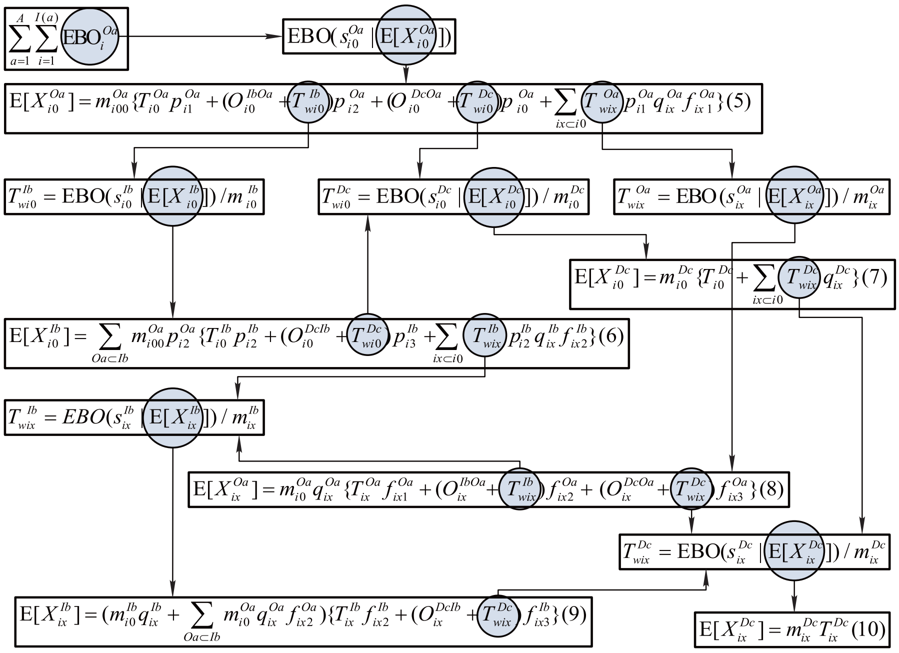


Fig. 2 Computing flow for MEMI objective function

$$m_{ix}^{Ib}=(m_{i0}^{Ib}q_{ix}^{Ib}+\sum_{Oa\subset Ib}m_{i0}^{Oa}q_{ix}^{Oa}f_{ix2}^{Oa})f_{ix2}^{Ib} \tag{15}$$

$$m_{i0}^{Dc}=\sum_{Oa\subset Dc}m_{i00}^{Oa}p_{i3}^{Oa}+\sum_{Ib\subset Dc}m_{i00}^{Oa}p_{i2}^{Oa}p_{i3}^{Ib} \tag{16}$$

$$m_{ix}^{Dc}=m_{i0}^{Dc}q_{ix}^{Dc}+\sum_{Oa\subset Dc}m_{i0}^{Oa}q_{ix}^{Oa}f_{ix3}^{Oa}+$$

$$\sum_{Ib\subset Dc}(m_{i0}^{Ib}q_{ix}^{Ib}+\sum_{Oa\subset Ib}m_{i0}^{Oa}q_{ix}^{Oa}f_{ix2}^{Oa})f_{ix3}^{Ib}. \tag{17}$$

During the optimization process, it needs to consider two constrains conditions: cost and repair level decision variable. As mentioned in Section 1, the costs mainly include the spares cost and the maintenance resource cost except spares. If we define C_i^S as the LRU total spares cost, C_i^R as the LRU total maintenance resource cost, c_{i0}^s as the LRU cost, c_{ix}^s as the SRU cost, c_{i0}^L as the LRU maintenance resource cost at the site $L(L\colon Oa, Ib, Dc)$, c_{ix}^L as the SRU maintenance resource cost at the site $L(L\colon Oa, Ib, Dc)$, and T as the support time (year), then we can get total spares and maintenance resource costs as

$$C_i^s=c_{i0}^s(\sum_{a=1}^{A}s_{i0}^{Oa}+\sum_{b=1}^{B}s_{i0}^{Ib}+\sum_{c=1}^{C}s_{i0}^{Dc})+$$

$$c_{ix}^s(\sum_{a=1}^{A}s_{ix}^{Oa}+\sum_{b=1}^{B}s_{ix}^{Ib}+\sum_{c=1}^{C}s_{ix}^{Dc}) \tag{18}$$

$$C_i^R=T\{\sum_{a=1}^{A}m_{i0}^{Oa}c_{i0}^{Oa}+\sum_{b=1}^{B}m_{i0}^{Ib}c_{i0}^{Ib}+\sum_{c=1}^{C}m_{i0}^{Dc}c_{i0}^{Dc}+$$

$$\sum_{a=1}^{A}m_{ix}^{Oa}c_{ix}^{Oa}+\sum_{b=1}^{B}m_{ix}^{Ib}c_{ix}^{Ib}+\sum_{c=1}^{C}m_{ix}^{Dc}c_{ix}^{Dc}\}. \tag{19}$$

Here the constraint condition is

$$\sum_{i=1}^{I}(C_i^S+C_i^R)\leqslant C_m. \tag{20}$$

The optimization model is used to determine the repair level, so each variable is a 0-1 variable, where 1 means maintenance at the site, and 0 means no maintenance at the site. The constraint conditions of repair level decision variables are as follows:

$$p_{i1}^{Oa},p_{i2}^{Oa},p_{i3}^{Oa},p_{i2}^{Ib},p_{i3}^{Ib}\in\{0,1\} \tag{21}$$

$$p_{i1}^{Oa}+p_{i2}^{Oa}+p_{i3}^{Oa}=1 \tag{22}$$

$$\begin{cases} p_{i2}^{Ib}+p_{i3}^{Ib}=1, & p_{i2}^{Oa}=1\\ p_{i2}^{Ib}+p_{i3}^{Ib}=0, & \text{else}\end{cases} \tag{23}$$

$$f_{ix1}^{Oa},f_{ix2}^{Oa},f_{ix3}^{Oa},f_{ix2}^{Ib},f_{ix3}^{Ib}\in\{0,1\} \tag{24}$$

$$f_{ix1}^{Oa}+f_{ix2}^{Oa}+f_{ix3}^{Oa}=1,\quad p_{i1}^{Oa}=1 \tag{25}$$

$$\begin{cases} f_{ix2}^{Ib}+f_{ix3}^{Ib}=1, & f_{ix2}^{Oa}=1\ \text{ or }\ p_{i2}^{Ib}=1\\ f_{ix2}^{Ib}+f_{ix3}^{Ib}=0, & \text{else}\end{cases}. \tag{26}$$

We define $\sum_{a=1}^{A}\sum_{i=1}^{I(a)}\mathrm{EBO}(s_{i0}^{Oa}|\mathrm{E}[X_{i0}^{Oa}])$ as the objective function, and we can get $\mathrm{EBO}(s_{i0}^{Oa}|\mathrm{E}[X_{i0}^{Oa}])$ according to the computing flow shown in Fig. 2. Equations (18) – (20) as cost constraint conditions, (21) – (26) as repair level decision variable constraint conditions, then we can get a MEMI LORA optimization model, as shown in (27):

$$\min \sum_{a=1}^{A}\sum_{i=1}^{I(a)} \mathrm{EBO}(s_{i0}^{Oa}|\mathrm{E}[X_{i0}^{Oa}])$$

$$\text{s.t.}\begin{cases} \sum_{i=1}^{I}(C_i^S + C_i^R) \leqslant C_m \\ p_{i1}^{Oa}, p_{i2}^{Oa}, p_{i3}^{Oa}, p_{i2}^{Ib}, p_{i3}^{Ib} \in \{0,1\}, \quad i=1,2,\dots,I \\ f_{ix1}^{Oa}, f_{ix2}^{Oa}, f_{ix3}^{Oa}, f_{ix2}^{Ib}, f_{ix3}^{Ib} \in \{0,1\}, \\ \quad x=1,2,\dots,X(i) \\ p_{i1}^{Oa}+p_{i2}^{Oa}+p_{i3}^{Oa}=1, \quad i=1,2,\dots,I \\ f_{ix1}^{Oa}+f_{ix2}^{Oa}+f_{ix3}^{Oa}=1, \quad \text{if } p_{i1}^{Oa}=1, \\ \quad x=1,2,\dots,X(i) \\ p_{i2}^{Ib}+p_{i3}^{Ib}=1, \quad \text{if } p_{i2}^{Oa}=1, i=1,2,\dots,I \\ p_{i2}^{Ib}+p_{i3}^{Ib}=0, \quad \text{if } p_{i2}^{Oa}=0, i=1,2,\dots,I \\ f_{ix2}^{Ib}+f_{ix3}^{Ib}=1, \quad \text{if } f_{ix2}^{Oa}=1 \text{ or } p_{i2}^{Ib}=1, \\ \quad x=1,2,\dots,X(i) \\ f_{ix2}^{Ib}+f_{ix3}^{Ib}=0, \quad \text{else}, \quad x=1,2,\dots,X(i) \end{cases}. \tag{27}$$

3. Optimization algorithm

3.1 Analysis of objective function EBO(s, p)

Different from the objective function $\mathrm{EBO}(s)$ in the traditional METRIC type model, we add a repair level decision variable p in our model, so inventories are not the only variable in the objective function $\mathrm{EBO}(s,p)$. As we know, the function $\mathrm{EBO}(s)$ is convex, while whether the function $\mathrm{EBO}(s,p)$ is convex or not still needs further study.

Now, we use the three-echelon, single-spare support organization for the object to verify the characteristic of the function.

Assuming there is only one site in each echelon, and the probability that the LRU is repaired at organization-level, intermediate-level or depot-level is $p=\{p_1,p_2,p_3\}$, then there are only three combinations for p: $p_1=\{1,0,0\}$, $p_2=\{0,1,0\}$, $p_3=\{0,0,1\}$. Assuming the spares stocks are zero at each echelon, the EBO is equal to the quantity demanded at each echelon, and the EBO at organization-level is

$$\begin{cases} \mathrm{EBO}(p_1)=mT_1 \\ \mathrm{EBO}(p_2)=m(O_2+T_2) \\ \mathrm{EBO}(p_3)=m(O_3+T_3) \end{cases}. \tag{28}$$

We define the maintenance time of organization-level, intermediate-level and depot-level as T_1, T_2, and T_3 respectively, and the delivery time of organization-level to intermediate-level, and intermediate-level to depot-level as O_2, and O_3 respectively. Assuming $\mathrm{EBO}(p_3) > \mathrm{EBO}(p_2) > \mathrm{EBO}(p_1)$, we can get

$$(O_3+T_3) > (O_2+T_2) > T_1. \tag{29}$$

The second difference formula $\mathrm{EBO}(p)$ is

$$\Delta^2\mathrm{EBO}(p) = \mathrm{EBO}(p_3) - 2\mathrm{EBO}(p_2) + \mathrm{EBO}(p_1) = m[O_3+T_3-2(O_2+T_2)+T_1]. \tag{30}$$

By adjusting the distance between sites or designing the appropriate maintenance time, we can make $\Delta^2\mathrm{EBO}(p) < 0$, so $\mathrm{EBO}(p)$ is non-convex. According to the convex optimization theory, $\mathrm{EBO}(s,p)$ needs to traverse the inventories (s) and repair level (p) respectively. However, $\mathrm{EBO}(p)$ is non-convex, and it will affect the application of the convex optimization algorithm which needs to be improved.

3.2 Algorithm

In Section 3.1, we have analyzed the objective function, next we should improve the algorithm to make $\mathrm{EBO}(s,p)$ convex. In this section, we firstly construct a convex function for non-convex function $\mathrm{EBO}(p)$, and then present the whole algorithm flow.

When traversing the repair level combination of all kinds of spares, the circulation of the repair level combination can be inserted in the circulation of the spares type. Therefore, if we have structured the EBO convex curve of all kinds of spares according to the repair level decision variables before traversing the spares types, all kinds of spares can be turned into convex optimization according to the margin iteration of the unit cost effect. Thus, we can conclude that the key step of solving the non-convex function optimization algorithm is to structure a convex function respectively for non-convex function $\mathrm{EBO}(p)$ of all kinds of spares.

The method of structuring convex function $\mathrm{EBO}(p)$-cost of all kinds of spares is the same as that of constructing the $\mathrm{EBO}(s)$-cost optimization curve, which enhances the unit cost effect of the certain spare to make an optimization decision. To a certain spare, there may be several repair levels to choose. Here, we fix the stocks $s$of $\mathrm{EBO}(s,p)$, and decide which kind of choice can make $\mathrm{EBO}(p)$ minimum. Based on this, we choose the minimum $\mathrm{EBO}(p)$ of repair levels in each step of increasing spares stocks, and get the optimization EBO(p)-cost convex function which is iterated in gradient direction.

We construct the EBO(p)-cost curve according to the theory of marginal analysis, and it is obvious that the curve is convex. Using the point on the EBO(p)-cost convex curve to make optimization analysis among all kinds of spares, we can get the optimization curve of EBO(s,p)-cost. The whole algorithm flow is shown in Fig. 3.

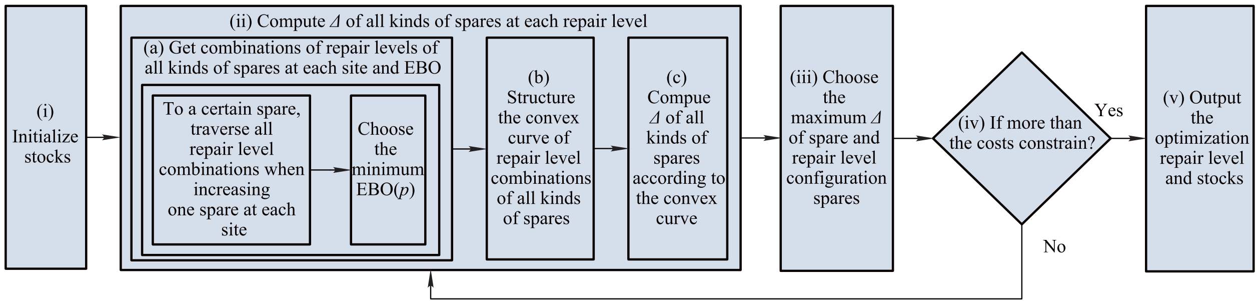


Fig. 3 Algorithm flow diagram

3.3 Method of improving optimization algorithm efficiency

In the optimization model joint LORA and spares stocks, the EBO is affected by stocks (s) and repair level (p). Since the repair level probability p and f are 0-1 variables, and p_{i1}^{Oa}, p_{i2}^{Oa}, p_{i3}^{Oa}, p_{i2}^{Ib}, p_{i3}^{Ib} and f_{ix1}^{Oa}, f_{ix2}^{Oa}, f_{ix3}^{Oa}, f_{ix2}^{Ib}, f_{ix3}^{Ib} can be obtained at corresponding sites, thus the number of repair level probability combinations at all sites in organization-level can reach $2^{(3A+2B)(1+X)}$. This means that when we compute Δ, the running speed of the algorithm will be affected seriously because of a large number of cycles of the EBO. However, in the algorithm iterative process, when we increase a certain type of spares, other types do not change with the increase of this type, which means there are many cases that can be disregarded when we compute Δ. Therefore, we list all possible combinations to improve the efficiency of the algorithm in Table 1.

Table 1 Relationship of values of the repair level probability

Site	p						f					
	No.	p_{i1}^{Oa}	p_{i2}^{Oa}	p_{i3}^{Oa}	p_{i2}^{Ib}	p_{i3}^{Ib}	No.	f_{ix1}^{Oa}	f_{ix2}^{Oa}	f_{ix3}^{Oa}	f_{ix2}^{Ib}	f_{ix3}^{Ib}
Organization-level	①	1	0	0	0	0	(a)	1	0	0	0	0
		1	0	0	0	0	(b)	0	1	0	1	0
		1	0	0	0	0	(c)	0	1	0	0	1
		1	0	0	0	0	(d)	0	0	1	0	0
Intermediate-level	②	0	1	0	1	0	(e)	0	0	0	1	0
		0	1	0	1	0	(f)	0	0	0	0	1
	③	0	1	0	0	1	(g)	0	0	0	0	0
Depot-level	④	0	0	1	0	0	(h)	0	0	0	0	0

Different combinations of repair level probability will lead to different EBO expressions, so it needs to compute the objective function value of all kinds of spares at each site when using the repair level.

Plus one LRU at each echelon site, we can get the number of combinations of each echlon site according to (5), (6) and (7), as shown in Table 2.

Table 2 Values of p and f when plus one LRU at each site

Site	p	f	Combinations	Total
Organization-level	①	(a-d)	4^X	$(4^X+2^X+1)^A$
	②	(e-f)	2^X	
	③	(g)	1	
	④	(h)	1	
Intermediate-level	②	(e-f)	2^X	$(2^X+1)^A$
	③	(g)	1	
Depot-level	③	(g)	1	2^A
	④	(h)	1	

Plus one SRU at each echelon site, we can get the number of combinations of each echelon site according to (8), (9) and (10), as shown in Table 3.

Table 3 Values of p and f when plus one SRU at each site

Site	p	f	Combinations	Total
Organization-level	①	(a-d)	4^X	$(4^X+2^X+1)^A$
	②	(e-f)	2^X	
	③	(g)	1	
	④	(h)	1	
Intermediate-level	②	(e-f)	2^X	$(2^X+1)^A$
	③	(g)	1	
Depot-level	③	(g)	1	2^A
	④	(h)	1	

4. Application case

Taking equipment maintenance planning as the background, we introduce how to apply the convex optimization algorithm based on the optimization theory proposed in this paper.

As shown in Fig.1(a), the support organization consists of three echelons: one depot-level site (D) supports two intermediate-level sites (I_1, I_2) which respectively support two organization-level sites (O_1, O_2, O_3, O_4). Each organization-level site supports ten systems, and these ten systems which contain eight types of LRUs are identical at the same site. Product structure is shown in Fig. 1(b).

(i) Input data

The uncertainty factors in the model are mainly demand rate of spares, unit cost and maintenance time. This case will present the influence of these factors on the repair level and the inventory with the simple variable method. As shown in Table 4, the maintenance time is the only difference between LRU_1 and LRU_2, the unit cost is the only difference between LRU_3 and LRU_4, the demand rate is the only difference between LRU_5 and LRU_6, and the data of LRU_7 and LRU_8 are different with the first six LRUs. The maintenance resource cost except spares in each repair level is shown in Table 5.

Table 4 Input date of multi-site multi-type LRUs

LRU	Demand rate/year	T_{i0}^{Dc}/year	T_{i0}^{Ib}/year	T_{i0}^{Oa}/year	T_{i0}^{DI}/year	T_{i0}^{DO}/year	T_{i0}^{IO}/year	Unit cost/ thousand dollars	Costs constrain/ thousand dollars
LRU_1	19.58	0.020	0.038 1	0.049 0	0.011 9	0.028 4	0.011 5	5	450
	14.85	0.020	0.038 1	0.058 9	0.011 9	0.049 1	0.012 4	5	
	10.36	0.020	0.036 0	0.057 5	0.012 1	0.048 8	0.021 9	5	
	17.43	0.020	0.036 0	0.055 5	0.012 1	0.040 4	0.024 0	5	
LRU_2	19.58	0.020	0.033 1	0.031 0	0.011 9	0.028 4	0.011 5	5	
	14.85	0.020	0.033 1	0.050 8	0.011 9	0.049 1	0.012 4	5	
	10.36	0.020	0.027 6	0.035 6	0.012 1	0.048 8	0.021 9	5	
	17.43	0.020	0.027 6	0.052 6	0.012 1	0.040 4	0.024 0	5	
LRU_3	15.06	0.020	0.030 0	0.047 6	0.011 9	0.028 4	0.011 5	4	
	11.39	0.020	0.030 0	0.051 0	0.011 9	0.049 1	0.012 4	4	
	13.50	0.020	0.023 0	0.037 6	0.012 1	0.048 8	0.021 9	4	
	13.52	0.020	0.023 0	0.035 9	0.012 1	0.040 4	0.024 0	4	
LRU_4	15.06	0.020	0.030 0	0.047 6	0.011 9	0.028 4	0.011 5	5	
	11.39	0.020	0.030 0	0.051 0	0.011 9	0.049 1	0.012 4	5	
	13.50	0.020	0.023 0	0.037 6	0.012 1	0.048 8	0.021 9	5	
	13.52	0.020	0.023 0	0.035 9	0.012 1	0.040 4	0.024 0	5	
LRU_5	15.38	0.020	0.035 0	0.036 9	0.011 9	0.028 4	0.011 5	6	
	19.62	0.020	0.035 0	0.059 9	0.011 9	0.049 1	0.012 4	6	
	18.00	0.020	0.020 1	0.056 1	0.012 1	0.048 8	0.021 9	6	
	11.46	0.020	0.020 1	0.042 9	0.012 1	0.040 4	0.024 0	6	
LRU_6	15.13	0.020	0.035 0	0.036 9	0.011 9	0.028 4	0.011 5	6	
	11.84	0.020	0.035 0	0.059 9	0.011 9	0.049 1	0.012 4	6	
	13.38	0.020	0.020 1	0.056 1	0.012 1	0.048 8	0.021 9	6	
	13.90	0.020	0.020 1	0.042 9	0.012 1	0.040 4	0.024 0	6	
LRU_7	12.35	0.020	0.028 1	0.058 3	0.011 9	0.028 4	0.011 5	7	
	11.69	0.020	0.028 1	0.040 6	0.011 9	0.049 1	0.012 4	7	
	11.89	0.020	0.033 0	0.043 5	0.012 1	0.048 8	0.021 9	7	
	17.80	0.020	0.033 0	0.050 6	0.012 1	0.040 4	0.024 0	7	
LRU_8	15.11	0.020	0.038 6	0.043 1	0.011 9	0.028 4	0.011 5	8	
	18.12	0.020	0.038 6	0.054 5	0.011 9	0.049 1	0.012 4	8	
	12.08	0.020	0.030 7	0.056 3	0.012 1	0.048 8	0.021 9	8	
	11.95	0.020	0.030 7	0.039 0	0.012 1	0.040 4	0.024 0	8	

Table 5 Maintenance resource cost of LRUs in each repair level (thousand dollars)

LRU	Depot D	Intermediate I_1I_2	Organization $O_1O_2O_3O_4$
LRU_1	1.81	2.91	3.55
LRU_2	1.39	2.71	3.10
LRU_3	1.12	2.34	3.26
LRU_4	1.83	2.92	3.38
LRU_5	1.69	2.08	3.83
LRU_6	1.14	2.55	3.35
LRU_7	1.24	2.13	3.06
LRU_8	1.08	2.49	3.51

(ii) Output

In the cost constrain of 450 thousand dollars, the total maintenance resource cost of all LRUs is 262.82 thousand dollars and the total spare cost is 187.18 thousand dollars, and the optimization value of the EBO is 5.064.

In the convex optimization algorithm, according to the principle of the highest unit cost-effectiveness, selecting sequence of spares based on marginal analysis is shown in Fig. 4, the Y-axis represents the type of LRU, while the X-axis represents selecting sequence, the length of which represents cost. The length of each segment represents unit cost and the label of which respectively represents the stocking site and selecting sequence of each LRU.

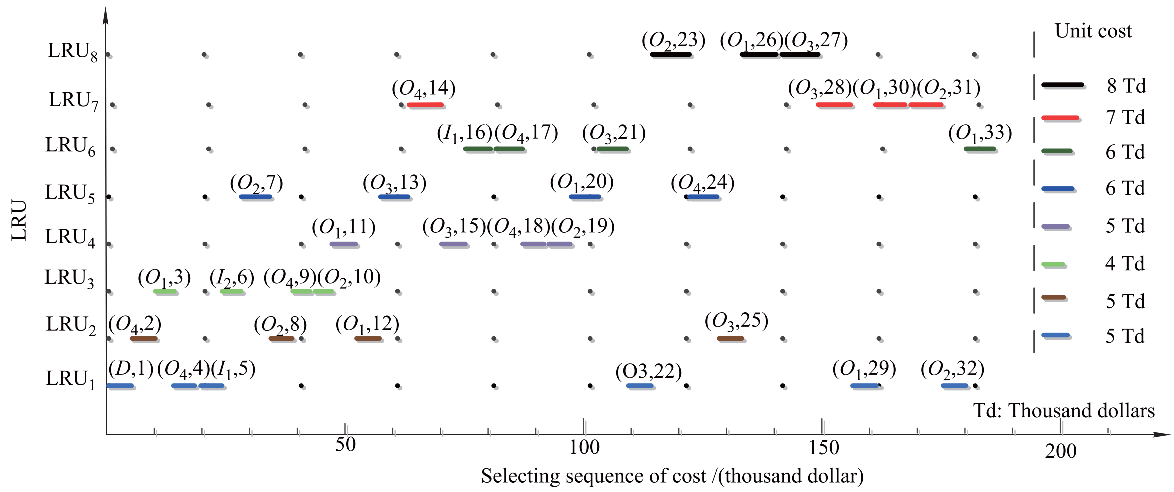


Fig. 4 Selecting sequence of spares

The optimal repair level of each LRU is shown in Table 6. Table 6(a) presents the optimal repair level of LRUs which are respectively deployed in site O_1 and site O_2 under the intermediate-level site I_1. Table 6(b) presents the optimal repair level of LRUs which are respectively deployed in site O_3 and site O_4 under the intermediate-level site I_2.

Table 6(a) Optimal repair level of each LRU

Intermediate-level(I_1)			
Organization-level(O_1)		Organization-level(O_2)	
Failure	Repair level	Failure	Repair level
LRU$_1$	D	LRU$_1$	D
LRU$_2$	O_1	LRU$_2$	D
LRU$_3$	I_1	LRU$_3$	I_1
LRU$_4$	I_1	LRU$_4$	I_1
LRU$_5$	O_1	LRU$_5$	D
LRU$_6$	D	LRU$_6$	D
LRU$_7$	I_1	LRU$_7$	I_1
LRU$_8$	O_1	LRU$_8$	D

Table 6(b) Optimal repair level of each LRU

Intermediate-level(I_2)			
Organization-level(O_3)		Organization-level(O_4)	
Failure	Repair level	Failure	Repair level
LRU$_1$	D	LRU$_1$	D
LRU$_2$	O_3	LRU$_2$	I_2
LRU$_3$	O_3	LRU$_3$	O_4
LRU$_4$	O_3	LRU$_4$	O_4
LRU$_5$	I_2	LRU$_5$	O_4
LRU$_6$	I_2	LRU$_6$	O_4
LRU$_7$	O_3	LRU$_7$	O_4
LRU$_8$	I_2	LRU$_8$	O_4

The optimal stocks allocation of each LRU is shown in Table 7.

Table 7 Stocks allocation of multi-site multi-type LRUs

LRU	Depot	Intermediate		Organization			
		I_1	I_2	$O_1(I_1)$	$O_2(I_1)$	$O_3(I_2)$	$O_4(I_2)$
LRU$_1$	1	1	0	1	1	1	1
LRU$_2$	0	0	0	1	1	1	1
LRU$_3$	0	0	1	1	1	0	1
LRU$_4$	0	0	0	1	1	1	1
LRU$_5$	0	0	0	1	1	1	1
LRU$_6$	0	1	0	1	0	1	1
LRU$_7$	0	0	0	1	1	1	1
LRU$_8$	0	0	0	1	1	1	0

Fig. 5 shows the EBO_i-cost optimal curve of each LRU and the total EBO-cost optimal curve, from which we can see the optimal EBO of eight LRUs in different total costs. In this case, the total EBO is 5.064 under spares cost of 187.18, and the red dot on the total EBO-cost curve is the optimal selecting dot by the marginal analysis which also can be seen in details in Fig. 4.

In order to verify our algorithm, we simulate the model with the optimal inventory allocation and repair level by the SIMLOX, which is a simulation software for logistic support systems. We take the result of the optimal inventory allocation and repair level under the cost constraint of 450 provided by our algorithm as input data, and let these forty systems carry out an identical mission for one year. Fig. 6 presents the simulation curve, from which it can be found that with the same input of Table 4, the mean EBO of the forty systems carrying out the identical mission for one year is 5.152.

(iii) Result analysis

According to the output, comparing with the results of LRU$_1$ and LRU$_2$, the maintenance time affects the combination of the repair level.

Comparing with the results of LRU_3 and LRU_4, the unit cost affects the stocks allocation at each site. Comparing with the results of LRU_5 and LRU_6, the demand rate of spares affects the stocks allocation at each site. Drawing a conclusion, from the results, we can get that the maintenance time of LRU at each site can affect the combination of the repair level, demand rate and unit cost of LRU, and affect stocks allocation at each site. In addition, according to the simulation result, comparing the EBO of theoretical calculation with that of the simulation, we can see that there is only little difference. Thus, the correctness of our algorithm can be confirmed.

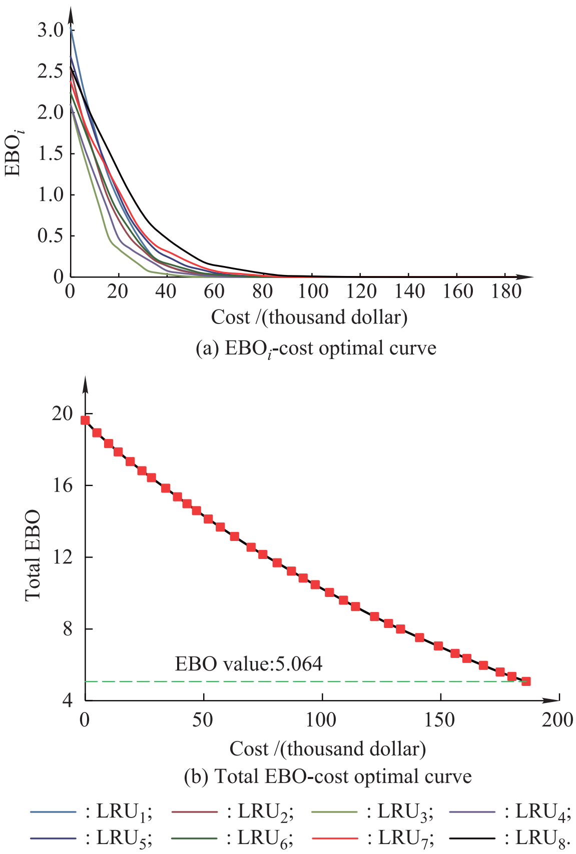


Fig. 5 Optimal curve of multi-type LRUs

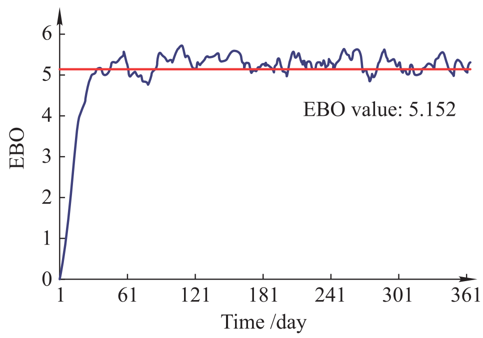


Fig. 6 EBO simulation

5. Conclusions and further research

In this paper, we propose a method of LORA considering the maintenance time in equipment support systems, analyze the problem of level of repair systematically, deduce the LORA objective function and give the modeling condition. Then, we provide the optimal objective function formula and constraint condition at the MEMI support organization, and build the MEMI optimization model joint LORA and spares stocks. In addition, we analyze the characteristics of the optimization problem, design a multi-variable convex optimization algorithm, explain the theory and flow of the algorithm, and provide solutions to the non-convex objective function and methods for improving the efficiency of the algorithm. Finally, we confirm the correctness of the proposed algorithm through a numerical example by virtue of simulation.

The contribution of the paper is that it analyzes the restrictive relation of the repair level and spare stock, and the transferring and coupling relations of the number of failures sent to repair, spares demand and spare backorders random variables in multi-indenture product systems. This paper introduces the maintenance time into the traditional quantitative repair level analysis and establishes the joint optimal model of the repair level and spare stock. Meanwhile, the support organization of multi-echelon multi-station in the model is asymmetric which means it is more truthful since the operation profile and failure of systems in all stations and the maintenance capability can be different. In addition, the paper designes the multi-variable convex optimization algorithm for the model, which can be highly effective when applied to complicated support systems, and verifies the correctness of the algorithm and the model by simulation.

In further research, we will consider all maintenance resources in our model, build optimization models of limited maintenance capability, and take the correlation of inter-depot maintenance resources into account.

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The capacitated facility location-allocation problem under uncertain environment

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Abstract. Facility location-allocation (FLA) problem has been widely studied by operational researchers due to its many practical applications. In real life, it is usually very hard to present the customers' demands in a precise way and thus they are regarded to be uncertain. Since the uncertain demands can be estimated from historical data, researchers tried to describe FLA problem under stochastic environment. Although stochastic models can cater for a variety of cases, they are not sufficient to describe many other situations, where the probability distribution of customers' demands may be unknown or partially known. Instead we have to invite some domain experts to evaluate their belief degree that each event will occur. This paper will consider the capacitated FLA problem under small sample or no-sample cases and establish an uncertain expected value model based on uncertain measure. In order to solve this model, the simplex algorithm, Monte Carlo simulation and a genetic algorithm are integrated to produce a hybrid intelligent algorithm. Finally, a numerical example is presented to illustrate the uncertain model and the algorithm.

Keywords: Location-allocation problem, uncertainty theory, uncertain measure, uncertain programming, genetic algorithm

1. Introduction

Facility location-allocation (FLA) problem, as one of the most critical and strategic issues in supply-chain design and management, exhibits a significant impact on market share and profitability. Besides supply-chain management, it is also widely used in practical life, such as building an emergency service systems and constructing a telecommunication networks, etc.

FLA problem was initially studied by Cooper [5] in 1963, which wants to decide locations of warehouses and allocation of customers demand given the locations and demand of customers. Since then, this problem has received much attention from other researchers and it has been analyzed in a number of different ways. Hakimi [13, 14] applied it in network design as a powerful tool. In 1982, Murtagh and Niwattisyawong [32] proposed the capacitated FLA problem, which is considered as one of the most important researches in this field, specially focusing on facilities which have capacity constraints. Many extensions of FLA problem were studied, Such as continuous site location problem Jiang and Yuan [16], joint FLA problem Jayaraman and Pirkul [15] and multi-objective FLA problem Revelle and Laporte [34]. For detailed review of literature on FLA problem, readers are referred to Klose and Drexl [17]. Megiddo and Supowit [30] have proved that the FLA problem is strongly NP-hard, and thus a large amount of solution approaches for different models have been proposed in the past decades Kuenne and Soland [18], Murray and Church [31]. A series of heuristic algorithms have also been developed to solve complicated FLA problems Ernst and Krishnamoorthy [9], Gong et al. [12].

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A limitation of most existing studies on FLA problem is that customers' demands are usually assumed deterministic and therefore a linear inventory holding cost is adopted. Without considering the uncertainty of customers' demands, those models usually lead to sub-optimal in terms of total cost. In recent years, some studies consider stochastic customers' demands and incorporate inventory policy into FLA problem Logendran and Terrell [19], Sabri and Beamon [35], Zhou and Liu [38, 41]. Although stochastic models can cater for a variety of cases, they are not sufficient to describe many other situations, where the probability distributions of customers' demands may be unknown or partially known. In order to have an approximate understanding of these cases, we usually consult the experts and obtain their empirical data. The empirical data from experts, like "about 100", "more than 200", etc., can be regarded as fuzzy variable initialized by Zadeh [39]. In the past decades many researchers have introduced fuzzy theory into FLA problem Bhattacharya et al. [1], Chen and Wei [3], Darzentas [7], Zhou and Liu [42, 43], Wen and Iwamura [37].

However, a lot of surveys showed that this assumption is also not suitable. For example, we say "the customer's demand is about 100". Generally, we employ fuzzy variable to describe the concept of "about 100", then there exists a membership function, such as a triangular one (90, 100, 110). Based on this membership function, possibility theory will conclude: (a) the demand is "exactly 100" with belief degree 1, and (b) the demand is "not 100" with belief degree 1. Obviously, the belief degree of "exactly 100" is almost zero. Besides, (b) indicates "not 100" and "exactly 100" have the same belief degree. These conclusions are improper. In order to have a better mathematical tool to deal with empirical data, uncertainty theory was founded by Liu [20] in 2007 and refined in 2010 [24]. In this paper, we will assume the customers's demands are uncertain variables, and propose an uncertain facility location-allocation model with capacitated supply.

The rest of this paper is organized as follows. Section 2 will introduce some basic concepts and properties about uncertain variables. An uncertain capacitated facility location-allocation model as well as its equivalent crisp model is presented in Section 3. In order to solve this uncertain model, we integrate the simplex algorithm, Monte Carlo simulation and genetic algorithm to design a powerful hybrid intelligent algorithm in Section 4. In Section 5, a numerical example will be provided to illustrate the performance and the effectiveness of the proposed model and algorithm. Finally, Section 6 will give the conclusion, in which the contributions of this paper and future research plans are shown.

2. Preliminaries

Uncertainty theory was founded by Liu [20] in 2007 and refined by Liu [24] in 2010. As extensions of uncertainty theory, uncertain process and uncertain differential equations Liu [21], uncertain calculus Liu [22] were proposed. Uncertain programming was first proposed by Liu [23] in 2009, which wants to deal with the optimal problems involving uncertain variable. This work was followed by an uncertain multiobjective programming and an uncertain goal programming Liu and Chen [26], and an uncertain multilevel programming Liu and Yao [27]. Since that, uncertainty theory was used to solve variety of real optimal problems, including finance Chen and Liu [4], Peng [33], Liu [28], reliability analysis Liu [25], Zeng et al. [40], graph Gao [10], Gao and Gao [11], et al. Nowadays uncertainty theory has become a branch of axiomatic mathematics for modeling human uncertainty. In this section, we will state some basic concepts and results on uncertain variable and uncertain programming. These results are crucial for the remainder of this paper.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\} \in [0, 1]$. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu [20][22] presented the four axioms:

Axiom 1. $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3. For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots, \infty$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}.$$

If the set function $\mathcal{M}$ satisfies the first three axioms, it is called an uncertain measure.

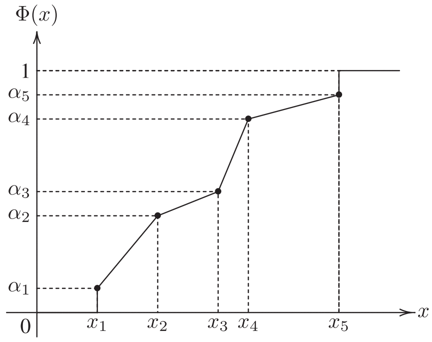

Fig. 1. Empirical uncertainty distribution.

Definition 1. Liu [20] Let Γ be a nonempty set, $\mathcal{M}$ a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

2.1. *Uncertain variable*

Definition 2. Liu [20] An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma\}|\xi(\gamma) \in B\} \tag{1}$$

is an event.

Definition 3. Liu [20] The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\} \tag{2}$$

for any real number x.

Example 1. An uncertain variable ξ is said to have an empirical uncertainty distribution if

$$\Phi(x) = \begin{cases} 0, & \text{if } x < x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1} \\ 1, & \text{if } x > x_n \end{cases}$$

denoted by $\mathcal{E}(x_1, \alpha_1, x_2, \alpha_2, \cdots, x_n, \alpha_n)$, where $x_1 < x_2 < \cdots < x_n$ and $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$.

Example 2. The inverse uncertainty distribution of zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \ge 0.5. \end{cases}$$

Definition 4. Liu [24] Let ξ be an uncertain variable. Then the expected value of ξ is defined by

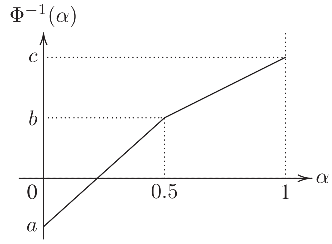

Fig. 2. Inverse zigzag uncertainty distribution.

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}\mathrm{d}r \tag{3}$$

provided that at least one of the two integrals is finite.

Example 3. The zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an expected value

$$E[\xi] = \frac{a + 2b + c}{4}.$$

Example 4. Let ξ have an empirical uncertainty distribution, i.e., $\xi \sim \mathcal{E}(x_1, \alpha_1, x_2, \alpha_2, \cdots, x_n, \alpha_n)$. Then

$$E[\xi] = \tfrac{\alpha_1+\alpha_2}{2}x_1 + \textstyle\sum_{i=2}^{n-1} \tfrac{\alpha_{i+1}-\alpha_{i-1}}{2} x_i$$
$$+ \left(1 - \tfrac{\alpha_{n-1}+\alpha_n}{2}\right) x_n$$

where $x_1 < x_2 < \cdots < x_m$ and $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$.

Definition 5. Liu [24] An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

Definition 6. Liu [22] The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \tag{4}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Theorem 1. *Liu and Ha [29] Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \tag{5}$$

provided that $E[\xi]$ exists.

Example 5. Let ξ and η be independent and nonnegative uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then

$$E[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)|\mathrm{d}\alpha.$$

Example 6. Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then

$$E[\exp(\xi)] = \int_0^1 \exp(\Phi^{-1}(\alpha))\mathrm{d}\alpha.$$

Theorem 2. *Liu [24] Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{6}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \tag{7}$$

Example 7. Let ξ be an uncertain variable with regular uncertainty distribution Φ. since $f(x) = ax + b$ is a strictly increasing function for any constants $a > 0$ and b, the inverse uncertainty distribution of $a\xi + b$ is

$$\Psi^{-1}(\alpha) = a\Phi_1^{-1}(\alpha) + b. \tag{8}$$

Example 8. Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Since

$$f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n \tag{9}$$

is a strictly increasing function, the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{10}$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha). \tag{11}$$

2.2. *Uncertain programming*

Uncertain programming, which was first proposed by Liu [23] in 2009, is a type of mathematical programming involving uncertain variables. After that, an uncertain multiobjective programming and an uncertain goal programming Liu and Chen [26], and an uncertain multilevel programming Liu and Yao [27] were provided.

Assume that x is a decision vector, and $\boldsymbol{\xi}$ is an uncertain vector. Since the uncertain programming model contains the uncertain objective function $f(x, \boldsymbol{\xi})$ and uncertain constraints $g_j(x, \xi) \leq 0$, $j = 1, 2, \cdots, p$, Liu [23] proposed the following uncertain programming model,

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(x, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(x, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p. \end{cases} \tag{12}$$

Definition 7. Liu [24] A vector x is called a feasible solution to Model (12) if

$$\mathcal{M}\{g_j(x, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j \tag{13}$$

for $j = 1, 2, \cdots, p$.

Definition 8. Liu [24] A feasible solution x^* is called an optimal solution to Model (12) if

$$E[f(x^*, \boldsymbol{\xi})] \leq E[f(x, \boldsymbol{\xi})] \tag{14}$$

for any feasible solution x.

Theorem 3. *Liu [24] Assume $f(x, \xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, and $g_j(x, \xi_1, \xi_2, \cdots, \xi_n)$ are strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \cdots, \xi_n$ for $j = 1, 2, \cdots, p$. If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the uncertain programming*

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(x, \xi_1, \xi_2, \cdots, \xi_n)] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \leq 0\} \geq \alpha_j, \\ \qquad\qquad j = 1, 2, \cdots, p \end{cases} \tag{15}$$

is equivalent to the crisp mathematical programming

$$\begin{cases} \min_{x} \int_0^1 f(x, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \\ \quad \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \\ \text{subject to:} \\ \quad g_j(x, \Phi_1^{-1}(\alpha_j), \cdots, \Phi_k^{-1}(\alpha_j), \\ \quad \Phi_{k+1}^{-1}(1-\alpha_j), \cdots, \Phi_n^{-1}(1-\alpha_j)) \leq 0, \\ \qquad\qquad j = 1, 2, \cdots, p. \end{cases}$$

3. Uncertain capacitated FLA problem

The capacitated continuous FLA problem is to find the locations of n facilities in continuous space in order to serve customers at m fixed points as well as the allocation of each customer to the facilities so that total transportation costs are minimized. In order to model the capacitated FLA problem, firstly we make some assumptions:

1. Each facility has a limited capacity. Thus we need to select locations and decide the amount from each facility i to each customer j.
2. The path between any customers and facilities is connected and transportation cost is proportionate to the quantity supplied and the travel distance.
3. Facility i is assumed to be located within a certain region $R_i = \{(x_i, y_i)|g_i(x_i, y_i) \leq 0\}$, $i = 1, 2, \ldots, n$, respectively.

As we know, when we use probability or statistics to build models, a large amount of historical data is needed. However, in most case the sample size is too small (even no-sample) to estimate a probability distribution. Then we have to invite some domain experts to evaluate their belief degree that each event will occur. This provides a motivation for Liu [20] to found an uncertainty theory. Then we will give the symbols and notations as follows:

$i = 1, 2, \ldots, n$ is the index of facilities;

$j = 1, 2, \ldots, m$ is the index of customers;

(a_j, b_j) denotes the location of customer j, $1 \leq j \leq m$;

ξ_j is the uncertain demand of customer j, $1 \leq j \leq m$;

Φ_j is the uncertainty distribution of ξ_j, $1 \leq j \leq m$;

s_i is the capacity of facility i, $1 \leq i \leq n$;

(x_i, y_i) is the decision variable which represents the location of facility i, $1 \leq i \leq n$;

z_{ij} denotes the quantity supplied by facility i to customer j, $1 \leq i \leq n$, $1 \leq j \leq m$.

For convenience, we also write

$$\mathbf{Z}(\boldsymbol{\xi}) = \left\{ z \,\middle|\, \begin{array}{l} z_{ij} \geq 0,\ i=1,2,\ldots,n,\ j=1,2,\ldots,m \\ \sum_{i=1}^{n} z_{ij} = \xi_j,\ j = 1, 2, \ldots, m \\ \sum_{j=1}^{m} z_{ij} \leq s_i,\ i = 1, 2, \ldots, n \end{array} \right\}. \tag{16}$$

Then we can give the uncertain transportation cost with the best allocation z,

$$C(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) = \min_{z \in \mathbf{Z}(\boldsymbol{\xi})} \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2}. \tag{17}$$

If $\mathbf{Z}(\boldsymbol{\xi})$ is an empty set for some $\boldsymbol{\xi}$, we can define

$$C(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) = \sum_{j=1}^{m} \max_{1 \leq i \leq n} \xi_j \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2}. \tag{18}$$

3.1. Uncertainty distributions of customers' demands

Liu [24] proposed a questionnaire survey for collecting expert's experimental data. It is based on expert's experimental data rather than historical data. The starting point is to invite one expert who is asked to complete a questionnaire about the meaning of an uncertain demand ξ like "How many is the customer's demand".

We first ask the domain expert to choose a possible value x that the uncertain demand ξ may take, and then quiz him

"How likely is ξ less than or equal to x?"

Denote the expert's belief degree by α. An expert's experimental data (x, α) thus acquired from the domain expert.

Repeating the above process, we can obtain the following expert's experimental data

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n) \tag{19}$$

that meet the following consistence condition (perhaps after a rearrangement)

$$x_1 < x_2 < \cdots < x_n,\ \ 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \alpha_n \leq 1. \tag{20}$$

Based on those expert's experimental data, Liu [24] suggested an empirical uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & x \le x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & x_i \le x \le x_{i+1}, \\ 1, & x > x_n. \end{cases} \tag{21}$$

denoted by $\mathcal{E}(x_1, \alpha_1, x_2, \alpha_2, \cdots, x_n, \alpha_n)$. Essentially, it is a type of linear interpolation method. The distribution function has been shown in Fig. 1.

Assume there are m domain experts and each produces an uncertainty distribution. Then we may get m uncertainty distributions $\Phi_1(x), \Phi_2(x), \cdots, \Phi_m(x)$. The Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. Wang et al. [36] recast the Delphi method as a process to determine the uncertainty distribution. The main steps are listed as follows:

Step 1: The m domain experts provide their expert's experimental data,

$$(x_{ij}, \alpha_{ij}), \ \ j = 1, 2, \cdots, n_i, \ i = 1, 2, \cdots, m. \tag{22}$$

Step 2: Use the i-th expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ to generate the i-th expert's uncertainty distribution Φ_i.

Step 3: Compute $\Phi(x) = w_1\Phi_1(x) + w_2\Phi_2(x) + \cdots + w_m\Phi_m(x)$ where $w_1, w_2, \cdots, w_m$ are convex combination coefficients.

Step 4: If $|\alpha_{ij} - \Phi(x_{ij})|$ are less than a given level $\varepsilon > 0$, then go to Step 5. Otherwise, the i-th expert receives the summary (Φ and reasons), and then provides a set of revised expert's experimental data. Go to Step 2.

Step 5: The last Φ is the uncertainty distribution of the customer's demand.

3.2. *Uncertain expected value model*

The essential idea of the expected cost minimization model is to optimize the expected value of $C(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ subject to some expected constraints. Zhou and Liu [41] have formulated the expected cost minimization model under stochastic demands. Here we shall give the model with uncertain environment as follows:

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} E[C(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})] \\ \text{subject to :} \\ \qquad g_i(\boldsymbol{x}, \boldsymbol{y}) \le 0, i = 1, 2, \ldots, p \end{cases} \tag{23}$$

where $g_i(\boldsymbol{x}, \boldsymbol{y}) \le 0, i = 1, 2, \ldots, p,$ are the potential regions of locations of new facilities.

Since the uncertain transportation cost $C(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$, the uncertain capacitated FLA model is equivalent to

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} \int_0^1 C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha))\mathrm{d}\alpha \\ \text{subject to :} \\ \qquad g_i(\boldsymbol{x}, \boldsymbol{y}) \le 0, i = 1, 2, \ldots, p. \end{cases} \tag{24}$$

The model is different from traditional programming models because there is a sub-optimal problem in it, i.e.,

$$C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha))$$

$$= \begin{cases} \min\limits_{z} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} z_{ij}\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \\ \text{subject to :} \\ \qquad z_{ij} \ge 0, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \\ \qquad \sum\limits_{i=1}^{n} z_{ij} = \Phi_j^{-1}(\alpha), \ j = 1, 2, \ldots, m \\ \qquad \sum\limits_{j=1}^{m} z_{ij} \le s_i, \ i = 1, 2, \ldots, n. \end{cases} \tag{25}$$

This sub-optimal problem can be easily solved by simplex algorithm because it is a linear programming.

4. Hybrid intelligent algorithm

Generally speaking, uncertain programming models are difficult to solve by traditional methods due to its complexity. Moreover the FlA problem has been proved to be NP-hard Megiddo and Supowit [30]. Heuristic methods have been shown to be the best way to tackle larger NP-hard problems. Modern heuristics such as simulated annealing, tabu search, genetic algorithms (GA), variable neighborhood search, and ant systems increase the chance of avoiding local optimality. In this paper, we use GA which was shown useful and effective in solving engineering design and optimization problems by numerous experiments to compute the FLA problem. And we use simplex algorithm to solve the sub-optimal problem (25) in uncertain expected value model. In this paper, we integrate the simplex algorithm, Monte Carlo simulation and genetic algorithm to produce a hybrid intelligent algorithm for solving the uncertain FLA model. We describe the algorithm as the following procedure:

Step 1. From the potential region $\{(\boldsymbol{x}, \boldsymbol{y})|g_i(\boldsymbol{x}, \boldsymbol{y}) \leq 0, i = 1, 2, \ldots, n\}$, initialize *pop_size* chromosomes $V_k = (\boldsymbol{x}^k, \boldsymbol{y}^k) = (x_1^k, x_2^k, \ldots, x_n^k, y_1^k, y_2^k, \ldots, y_n^k)$, $k = 1, 2, \ldots, pop\_size$, which denote the locations of all the facilities.

Step 2. Calculate the objective values U^k for all chromosomes $V_k, k = 1, 2, \ldots, pop\_size$ by Monte Carlo simulation, where the simplex algorithm is used to solve (25) to get the optimal cost $C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha))$.

Step 3. Compute the fitness of all chromosomes $V_k, k = 1, 2, \ldots, pop\_size$. The rank-based evaluation function is defined as

$$Eval(V_k) = \beta(1 - \beta)^{k-1}, \quad k = 1, 2, \ldots, pop\_size \tag{26}$$

where the chromosomes $V_1, V_2, \ldots, V_{pop\_size}$ are assumed to have been rearranged from good to bad according to their objective values U^k and $\beta \in (0, 1)$ is a parameter in the genetic system.

Step 4. Select the chromosomes for a new population. The selection process is based on spinning the roulette wheel characterized by the fitness of all chromosomes for *pop_size* times, and each time we select a single chromosome. Thus we obtain *pop_size* chromosomes, denoted also by V_k, $k = 1, 2, \ldots, pop\_size$.

Step 5. Renew the chromosomes $V_k, k = 1, 2, \ldots, pop\_size$ by crossover operation. We define a parameter P_c of a genetic system as the probability of crossover. This probability gives us the expected number $P_c \cdot pop\_size$ chromosomes undergoing the crossover operation.

Step 6. Update the chromosomes $V_k, k = 1, 2, \ldots, pop\_size$ by mutation operation. The parameter P_m is the probability of mutation, which gives us the expected number of $P_m \cdot pop\_size$ chromosomes undergoing the mutation operations.

Step 7. Repeat the second to the sixth steps for a given number of cycles.

Step 8. Report the best chromosome $V^* = (\boldsymbol{x}^*, \boldsymbol{y}^*)$ as the optimal locations.

5. A numerical example

Consider a company that wishes to locate four new facilities within the square region $[0, 100] \times [0, 100]$. Assume that there are 20 customers whose demands ξ_i are zigzag uncertainty variables. The location (a_i, b_i) of the customer i is given in Table 1, $i = 1, 2, \ldots, 20$. The capacities s_i of the four facilities are 100, 110, 120 and 130, respectively.

We choose 3 experts to give questionnaire surveys. The process is as follows:

Q1: May I ask you how many demand of customer 1 is? What do you think is the minimum demand?
A1: 13. (an expert's experimental data (13,0) is acquired).
Q2: What do you think the maximum demand is?
A2: 17. (an expert's experimental data (17,1) is acquired).
Q3: What do you think a likely demand is?
A3: 15.
Q4: What is the belief degree that the real demand is less than 15?
A4: 0.6. (an expert's experimental data (15,0.6) is acquired).

Every expert will give the questionnaire surveys to every customer. After all the surveys, all the customers' demands are shown in Table 2.

In the Delphi method, we set $w = 1/3$. Then we get the distributions of all the customers in Table 3.

In this example, Model (24) can be written as

Table 1
Locations of the customers

Customer j	Location	Customer j	Location
1	(28,42)	11	(14,78)
2	(18,50)	12	(90,36)
3	(74,34)	13	(78,20)
4	(74,6)	14	(24,52)
5	(70,18)	15	(54,6)
6	(72,98)	16	(62,60)
7	(60,50)	17	(98,14)
8	(36,40)	18	(36,58)
9	(12,4)	19	(38,88)
10	(18,20)	20	(32,54)

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} \int_0^1 C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha))\mathrm{d}\alpha \\ \text{subject to :} \\ \qquad 0 \le x_i \le 100, i = 1, 2, 3, 4 \\ \qquad 0 \le y_i \le 100, i = 1, 2, 3, 4 \end{cases} \tag{27}$$

where

$$C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha)) =$$

$$\begin{cases} \min\limits_{z} \sum_{i=1}^{4} \sum_{j=1}^{20} z_{ij}\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \\ \text{subject to :} \\ \qquad z_{ij} \ge 0, \quad i = 1, 2, 3, 4, \quad j = 1, 2, \ldots, 20 \end{cases}$$

$$\begin{cases} \sum_{i=1}^{4} z_{ij} = \Phi_j^{-1}(\alpha), \quad j = 1, 2, \ldots, 20 \\ \sum_{j=1}^{20} z_{1j} \le 100 \\ \sum_{j=1}^{20} z_{2j} \le 110 \\ \sum_{j=1}^{20} z_{3j} \le 120 \\ \sum_{j=1}^{20} z_{4j} \le 130. \end{cases} \tag{28}$$

In Model (27), the goal wants to minimize the total costs and the constraints denote the square region $[0, 100] \times [0, 100]$. In order to get the goal function $C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha))$, model (28) is given. In model (28), the constraint $\sum_{i=1}^{4} z_{ij} = \Phi_j^{-1}(\alpha)$ implies that the total supply amounts can not exceed

Table 2
Uncertain demands got from 3 experts

Customer j	Expert 1	Expert 2	Expert 3
1	$\mathcal{E}(13, 0, 15, 0.6, 17, 1)$	$\mathcal{E}(13, 0.15, 15, 0.75, 17, 1)$	$\mathcal{E}(13, 0, 15, 0.45, 17, 0.9)$
2	$\mathcal{E}(13, 0, 14, 0.5, 18, 1)$	$\mathcal{E}(13, 0, 14, 0.4, 18, 0.9)$	$\mathcal{E}(13, 0.1, 14, 0.6, 18, 1)$
3	$\mathcal{E}(12, 0.1, 14, 0.6, 16, 1)$	$\mathcal{E}(12, 0, 14, 0.4, 16, 0.8)$	$\mathcal{E}(12, 0, 14, 0.5, 16, 0.9)$
4	$\mathcal{E}(17, 0, 18, 0.4, 20, 0.7)$	$\mathcal{E}(17, 0, 18, 0.3, 20, 0.7)$	$\mathcal{E}(17, 0, 18, 0.5, 20, 1)$
5	$\mathcal{E}(21, 0, 23, 0.3, 26, 0.7)$	$\mathcal{E}(21, 0, 23, 0.4, 26, 0.8)$	$\mathcal{E}(21, 0, 23, 0.3, 26, 0.8)$
6	$\mathcal{E}(24, 0.1, 26, 0.7, 28, 1)$	$\mathcal{E}(24, 0, 26, 0.5, 28, 0.9)$	$\mathcal{E}(24, 0.1, 26, 0.6, 28, 1)$
7	$\mathcal{E}(13, 0, 15, 0.7, 16, 1)$	$\mathcal{E}(13, 0, 15, 0.6, 16, 1)$	$\mathcal{E}(13, 0.1, 15, 0.8, 16, 1)$
8	$\mathcal{E}(12, 0, 14, 0.3, 17, 0.8)$	$\mathcal{E}(12, 0, 14, 0.4, 17, 0.8)$	$\mathcal{E}(12, 0, 14, 0.5, 17, 1)$
9	$\mathcal{E}(13, 0, 15, 0.6, 17, 1)$	$\mathcal{E}(13, 0, 15, 0.5, 17, 1)$	$\mathcal{E}(13, 0.3, 15, 0.7, 17, 1)$
10	$\mathcal{E}(22, 0.1, 24, 0.3, 26, 0.6)$	$\mathcal{E}(22, 0, 24, 0.3, 26, 0.7)$	$\mathcal{E}(22, 0, 24, 0.2, 26, 0.5)$
11	$\mathcal{E}(13, 0, 15, 0.7, 17, 1)$	$\mathcal{E}(13, 0, 15, 0.5, 17, 1)$	$\mathcal{E}(13, 0, 15, 0.4, 17, 0.8)$
12	$\mathcal{E}(11, 0, 14, 0.6, 17, 1)$	$\mathcal{E}(11, 0, 14, 0.5, 17, 1)$	$\mathcal{E}(11, 0, 14, 0.4, 17, 0.9)$
13	$\mathcal{E}(13, 0.2, 15, 0.6, 19, 1)$	$\mathcal{E}(13, 0.3, 15, 0.8, 19, 1)$	$\mathcal{E}(13, 0, 15, 0.5, 19, 1)$
14	$\mathcal{E}(11, 0, 13, 0.2, 16, 0.5)$	$\mathcal{E}(11, 0, 13, 0.3, 16, 0.7)$	$\mathcal{E}(11, 0, 13, 0.4, 16, 0.9)$
15	$\mathcal{E}(20, 0, 24, 0.5, 26, 1)$	$\mathcal{E}(20, 0, 24, 0.4, 26, 0.9)$	$\mathcal{E}(20, 0, 24, 0.4, 26, 0.8)$
16	$\mathcal{E}(16, 0, 18, 0.3, 23, 1)$	$\mathcal{E}(16, 0, 18, 0.2, 23, 0.9)$	$\mathcal{E}(16, 0, 18, 0.4, 23, 1)$
17	$\mathcal{E}(18, 0, 19, 0.1, 22, 0.8)$	$\mathcal{E}(18, 0, 19, 0.3, 22, 0.8)$	$\mathcal{E}(18, 0, 19, 0.4, 22, 1)$
18	$\mathcal{E}(13, 0.2, 14, 0.3, 17, 0.6)$	$\mathcal{E}(13, 0.3, 14, 0.4, 17, 0.8)$	$\mathcal{E}(13, 0.1, 14, 0.2, 17, 0.7)$
19	$\mathcal{E}(16, 0, 1, 17, 0.3, 20, 1)$	$\mathcal{E}(16, 0, 2, 17, 0.4, 20, 1)$	$\mathcal{E}(16, 0, 1, 17, 0.2, 20, 0.8)$
20	$\mathcal{E}(19, 0, 22, 0.7, 25, 1)$	$\mathcal{E}(19, 0, 22, 0.5, 25, 0.8)$	$\mathcal{E}(19, 0, 22, 0.7, 25, 1)$

Table 3
Uncertain demands of the customers

Customer j	Expert 1	Expert 2	Expert 3
1	$\mathcal{E}(13, 0.05, 15, 0.6, 17, 0.96)$	11	$\mathcal{E}(13, 0, 15, 0.53, 17, 0.93)$
2	$\mathcal{E}(13, 0.03, 14, 0.5, 18, 0.96)$	12	$\mathcal{E}(11, 0, 14, 0.5, 17, 0.97)$
3	$\mathcal{E}(12, 0.03, 14, 0.5, 16, 0.9)$	13	$\mathcal{E}(13, 0.17, 15, 0.63, 19, 1)$
4	$\mathcal{E}(17, 0, 18, 0.4, 20, 0.8)$	14	$\mathcal{E}(11, 0, 13, 0.3, 16, 0.7)$
5	$\mathcal{E}(21, 0, 23, 0.3, 26, 0.77)$	15	$\mathcal{E}(20, 0, 24, 0.43, 26, 0.9)$
6	$\mathcal{E}(24, 0.37, 26, 0.6, 28, 0.97)$	16	$\mathcal{E}(16, 0, 18, 0.3, 23, 0.97)$
7	$\mathcal{E}(13, 0.03, 15, 0.7, 16, 1)$	17	$\mathcal{E}(18, 0, 19, 0.27, 22, 0.87)$
8	$\mathcal{E}(12, 0, 14, 0.4, 17, 0.93)$	18	$\mathcal{E}(13, 0.2, 14, 0.3, 17, 0.7)$
9	$\mathcal{E}(13, 0.1, 15, 0.6, 17, 1)$	19	$\mathcal{E}(16, 0, 13, 17, 0.3, 20, 0.93)$
10	$\mathcal{E}(22, 0.03, 24, 0.27, 26, 0.6)$	20	$\mathcal{E}(19, 0, 22, 0.63, 25, 0.93)$

Table 4
Comparison solutions

	pop_size	P_c	P_m	β	Locations	Cost	Error (%)
1	20	0.3	0.1	0.10	(72,15),(40,88),(60,51),(26,42)	5731	0.5
2	20	0.3	0.1	0.05	(78,19),(23,15),(64,77),(32,51)	5700	0.0
3	20	0.3	0.2	0.10	(74,20),(20,24),(60,91),(33,57)	5792	1.6
4	20	0.3	0.2	0.05	(73,16),(40,88),(60,51),(26,42)	5742	0.7
5	20	0.2	0.2	0.08	(75,21),(60,91),(60,24),(33,57)	5788	1.5
6	30	0.3	0.1	0.05	(72,14),(64,77),(23,15),(32,51)	5743	0.8
7	30	0.3	0.1	0.10	(75,17),(60,91),(20,24),(33,57)	5786	0.1
8	30	0.3	0.2	0.05	(74,19),(40,88),(60,51),(26,42)	5729	1.5
9	30	0.3	0.2	0.10	(78,19),(64,77),(23,15),(32,51)	5744	0.8
10	30	0.2	0.2	0.08	(76,17),(64,77),(23,15),(32,51)	5758	1.0

customer demand, $j = 1, 2, \ldots, 20$. Otherwise, the waste of resources appears, which is not allowed in supply chain. The last four constraints express the capacities of the four facilities.

In order to solve Model (27), the hybrid intelligent algorithm is employed. Monte Carlo simulation has been run with 5000 cycles in every generation in GA. After 1000 generations in GA, we get the optimal results in Table 4. In Table 4, we compare solutions with different number of chromosomes *pop_size*, different probability of crossover P_c and different probability of mutation P_m taken with the same stopping rule. It appears that all the minimal costs differ little from each other. The percent error of the last column can be expressed by (actual value - optimal value)/optimal value $\times 100\%$, where optimal value is the least one of all the ten minimal costs. It follows from Table 5 that the percent error does not exceed 1.6% when different parameters are selected, which implies that the hybrid intelligent algorithm is robust to the parameter settings and effective to solve Model (23).

6. Conclusion

In this paper, we have contributed to the research area of the FLA problem in the following three aspects.

(i) The expected value FLA model was proposed under uncertain environment, which can be converted to a crisp mathematical programming;
(ii) To solve the models efficiently, we integrated the simplex algorithm, Monte Carlo simulation and genetic algorithm to produce a hybrid intelligent algorithm.
(iii) A numerical example was provided to illustrate the expected value model and the performance of the hybrid intelligent algorithm.

Several further studies should be considered. For one thing, we will try to present other uncertain FLA models besides expected value model. In addition, we will try to apply the expected value FLA model to the actual case, and then modify this uncertain model. Moreover, we will give some researches to the hybrid FLA problem in which the randomness and fuzziness of the customers' demands are mixed up with each other.

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The α-cost minimization model for capacitated facility location-allocation problem with uncertain demands

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Abstract Facility location-allocation problem aims at determining the locations of some facilities to serve a set of spatially distributed customers and the allocation of each customer to the facilities such that the total transportation cost is minimized. In real life, the facility location-allocation problem often comes with uncertainty for lack of the information about the customers' demands. Within the framework of uncertainty theory, this paper proposes an uncertain facility location-allocation model by means of chance-constraints, in which the customers' demands are assumed to be uncertain variables. An equivalent crisp model is obtained via the α-optimistic criterion of the total transportation cost. Besides, a hybrid intelligent algorithm is designed to solve the uncertain facility location-allocation problem, and its viability and effectiveness are illustrated by a numerical example.

Keywords Location-allocation problem · Uncertainty theory · Uncertain variable · Genetic algorithm

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1 Introduction

With the rapid development of supply chain management in recent decades, facility location-allocation problem, one of the most representative problems in strategic supply chain management, has also received much attention from the researchers. Facility location-allocation problem was initialized by Cooper (1963) to find the locations of some facilities to serve some spatially distributed customers. In Murtagh and Niwattisyawong (1982), considered the capacity constraint of each facility, and proposed a capacitated facility location-allocation model. Then Badri (1999) and Hodey et al. (1997) studied the facility location-allocation problem using goal programming and multi-object programming approaches, respectively. In Murray and Church (1996), solved the facility location-allocation problem by simulated annealing algorithm. After that, Gong et al. (1995) solved the problem by genetic algorithm.

For lack of the precise demands of the customers, the facility location-allocation problem often contains some indeterminacy factors. In Logendran and Terrell (1988), presented an uncapacitated stochastic facility location-allocation model with random demands which are sensitive with respect to the price. Then Zhou and Liu (2003) proposed a capacitated facility location-allocation model with random demands. Darzentas (1987) first proposed a fuzzy facility location-allocation model in 1987, where the locations of the facilities were chosen from a set of discrete points. Then Zhou and Liu (2007) presented a facility location-allocation model, in which the customers' demands were described as fuzzy variables. After that, Wen and Iwamura (2008) gave a fuzzy facility location-allocation model via Hurwicz criterion.

In Liu (2007), founded an uncertainty theory to deal with human's belief degree mathematically, and in 2010, Liu (2010) refined it based on normality, duality, subadditivity and product measure axioms. As we know, in order to obtain the probability distribution of an indeterminacy quantity, we need a lot of samples to apply the statistics inference approach. However, due to economical or technological reasons, we sometimes have no sample about the indeterminacy quantity. In this case, we have to invite the domain experts to evaluate the belief degree that each possible event happens. Since human tends to overweight unlikely events (Kahneman and Tversky 1979), the belief degree has a much larger variance than the frequency, and cannot be treated as a probability distribution of a random variable. In this case, we can regard the belief degree as an uncertainty distribution of some uncertain variable, and deal with it via uncertainty theory. In order to model the evolution of uncertain phenomena, a concept of uncertain process was proposed by Liu (2008), as a generalization of uncertain variable.

Uncertain programming, as a spectrum of mathematical programming involving uncertain variables, was proposed by Liu (2009). Then Liu and Yao (2012) further studied uncertain multilevel programming, and Liu and Chen (2013) further studied uncertain multi-objective programming. As an application of uncertain programming, Gao (2012) proposed a facility allocation problem on a network. Qin and Kar (2013) presented a single-period inventory problem with uncertain demands, and Wang et al. (2013) presented an uncertain price discrimination model in labor market.

In this paper, we will assume the customers' demands are uncertain variables, and propose an uncertain facility location-allocation model with capacitated supply. The rest of this paper is organized as follows. Section 2 will introduce some basic concepts and properties about uncertain variables. Then an uncertain facility location-allocation model called α-cost minimization model as well as its equivalent crisp model will be presented in Sect. 3. In order to solve the uncertain model, a hybrid intelligent algorithm will be designed in Sect. 4, and a numerical example will be given to illustrate the algorithm in Sect. 6.

2 Preliminaries

Uncertainty theory was founded by Liu (2007) and refined by Liu (2010) to deal with human's belief degree. A concept of uncertain measure was first proposed to indicate the belief degree that an uncertain event happens. Then an uncertain variable was given to model the quantity in uncertain situation. After that, concepts of uncertainty distribution, expected value and variance were given to describe an uncertain variable. So far, uncertainty theory has been applied to risk analysis (Liu 2010), reliability analysis (Liu 2010; Zeng et al. 2013), logic (Li and Liu 2009).

Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. In order to assign a number $\mathcal{M}\{\Lambda\} \in [0, 1]$ to each event Λ, Liu (2007, 2009) presented four axioms:

Axiom 1. $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3. For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 1 (Liu 2007) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 2 (Liu 2009) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 3 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x.

Example 1 An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ \dfrac{x-a}{2(b-a)}, & \text{if } a < x \le b \\ \dfrac{x+c-2b}{2(c-b)}, & \text{if } b < x \le c \\ 1, & \text{if } x > c \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$ where a, b, c are real numbers with $a < b < c$.

Definition 4 (Liu 2010) An uncertainty distribution Φ of an uncertain variable ξ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. In this case, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Example 2 The inverse uncertainty distribution of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha \le 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha > 0.5. \end{cases}$$

Theorem 1 (Liu 2010) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

is an uncertain variable with an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

Example 3 Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Then

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n$$

is an uncertain variable with an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha).$$

Definition 5 (Liu 2007) The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite.

Theorem 2 *Let ξ be an uncertain variable with an uncertainty distribution Φ. If $E[\xi]$ exists, then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x.$$

Example 4 The expected value of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$E[\xi] = \frac{a + 2b + c}{4}.$$

Theorem 3 (Liu and Ha 2010) *Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right) \mathrm{d}\alpha.$$

3 Uncertain facility location-allocation problem

The capacitated continuous facility location-allocation problem is to find the locations of n facilities in a continuous space to serve m customers at some fixed points, and to allocate each customer to the facilities so that the total transportation cost is minimized. Before modelling the capacitated facility location-allocation problem, we first make some assumptions as follows.

1. Each facility has a limited capacity. Thus we need to decide the amount transported from each facility to each customer.
2. The path between every customer and every facility is connected, and the transportation cost is proportionate to the transportation distance and the transportation quantity.
3. The demand of each customer must be satisfied.
4. Facility i is assumed to be located within a certain region $R_i = \{(x_i, y_i)|g_i(x_i, y_i) \leq 0\}, i = 1, 2, \ldots, n$, respectively.

Sometimes, we have little sample data about the customers' demands in daily life. In this case, we have to invite the domain experts to evaluate their belief degree about the customers' demands. Now, we regard the belief degree about each customer's demand as an uncertainty distribution in the framework of uncertainty theory, and formulate an uncertain capacitated facility location-allocation model. We will use the following symbols in the model:

(a_j, b_j): location of customer $j,\ j = 1, 2, \ldots, m$;
ξ_j: demand of customer $j,\ j = 1, 2, \ldots, m$, uncertain variables;
Φ_j: uncertainty distribution of $\xi_j,\ j = 1, 2, \ldots, m$;
(x_i, y_i): location of facility $i,\ i = 1, 2, \ldots, n$, decision variables;
s_i: capacity of facility $i,\ i = 1, 2, \ldots, n$;
z_{ij}: quantity transported from facility i to customer j, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Since the uncertain demand of each customer must be satisfied, the total transportation cost C is an uncertain variable decided by

$$\begin{cases} C\left(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}(\gamma)\right) = \min\limits_{z_{ij}(\gamma)} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} z_{ij}(\gamma)\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \\ \text{subject to:} \\ \qquad \sum\limits_{i=1}^{n} z_{ij}(\gamma) \geq \xi_j(\gamma), \quad j = 1, 2, \ldots, m \\ \qquad \sum\limits_{j=1}^{m} z_{ij}(\gamma) \leq s_i, \quad i = 1, 2, \ldots, n \\ \qquad z_{ij}(\gamma) \geq 0, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m. \end{cases} \tag{1}$$

Chance-constrained programming, initialized by Charnes and Cooper (1961), is a powerful tool to deal with an indeterminacy system. The essence of chance-constrained programming is to optimize some critical value with a given confidence level subject to some chance constraints. Now we apply the chance-constrained programming model to facility location-allocation problem, and present an uncertain facility location-allocation model named α-cost minimization model,

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} f \\ \text{subject to :} \\ \qquad \mathcal{M}\{C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi}) \le f\} \ge \alpha \\ \qquad g_i(x_i, y_i) \le 0, \quad i = 1, 2, \dots, n \end{cases} \tag{2}$$

where $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})$ is the uncertain transportation cost determined by (1), and $\alpha \in (0, 1)$ is a predetermined confidence level.

Note that the α-cost minimization model (2) contains a sub-optimization problem (1), which enormously increases the complexity of the solution procedure. In order to design a hybrid intelligent algorithm to solve it, we first transform it to an equivalent crisp model.

Theorem 4 *The α-cost minimization model (2) is equivalent to the crisp model*

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} z_{ij}\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \\ \textit{subject to:} \\ \quad \sum\limits_{i=1}^{n} z_{ij} \ge \Phi_j^{-1}(\alpha), \quad j = 1, 2, \dots, m \\ \quad \sum\limits_{j=1}^{m} z_{ij} \le s_i, \quad i = 1, 2, \dots, n \\ \quad z_{ij} \ge 0, \quad i = 1, 2, \dots, n, \ j = 1, 2, \dots, m \\ \quad g_i(x_i, y_i) \le 0, \quad i = 1, 2, \dots, n. \end{cases} \tag{3}$$

Proof Since the uncertain transportation cost $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$, it follows from Theorem 1 that $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})$ has an inverse uncertainty distribution

$$\Phi^{-1}(\boldsymbol{x},\boldsymbol{y},\alpha) = C\left(\boldsymbol{x},\boldsymbol{y},\Phi_1^{-1}(\alpha),\Phi_2^{-1}(\alpha),\dots,\Phi_n^{-1}(\alpha)\right).$$

Thus

$$\mathcal{M}\{C(\boldsymbol{x},\boldsymbol{y},\xi_1,\xi_2,\dots,\xi_m) \le f\} \ge \alpha$$

holds if and only if

$$C\left(\boldsymbol{x},\boldsymbol{y},\Phi_1^{-1}(\alpha),\Phi_2^{-1}(\alpha),\dots,\Phi_n^{-1}(\alpha)\right) \le f.$$

As a result, the α-cost minimization model (2) is equivalent to

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} f \\ \text{subject to :} \\ \qquad C\left(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right) \leq f \\ \qquad g_i(x_i, y_i) \leq 0, \quad i = 1, 2, \ldots, n, \end{cases}$$

which can be rewritten as

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y}} C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) \\ \text{subject to :} \\ \qquad g_i(x_i, y_i) \leq 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{4}$$

Note that the cost function $C(\boldsymbol{x}, \boldsymbol{y}, \Phi_1^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))$ is determined by the following model

$$\begin{cases} \min\limits_{\boldsymbol{z}} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} z_{ij} \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \\ \text{subject to:} \\ \qquad \sum\limits_{i=1}^{n} z_{ij} \geq \Phi_j^{-1}(\alpha), \quad j = 1, 2, \ldots, m \\ \qquad \sum\limits_{j=1}^{m} z_{ij} \leq s_i, \quad i = 1, 2, \ldots, n \\ \qquad z_{ij} \geq 0, \quad i = 1, 2, \cdots, n, \ j = 1, 2, \ldots, m. \end{cases} \tag{5}$$

Combining Models (4) and (5), we have

$$\begin{cases} \min\limits_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} z_{ij} \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \\ \text{subject to:} \\ \qquad \sum\limits_{i=1}^{n} z_{ij} \geq \Phi_j^{-1}(\alpha), \quad j = 1, 2, \ldots, m \\ \qquad \sum\limits_{j=1}^{m} z_{ij} \leq s_i, \quad i = 1, 2, \ldots, n \\ \qquad z_{ij} \geq 0, \quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \\ \qquad g_i(x_i, y_i) \leq 0, \quad i = 1, 2, \ldots, n. \end{cases}$$

The theorem is thus proved.

4 Hybrid intelligent algorithm

Genetic algorithm has shown great effectiveness in solving engineering design and optimization problems in numerous experiments. In this section, we will employ simplex method to solve the sub-optimal problem (5), and genetic algorithm to solve the α-cost minimization problem (4). In other words, we will integrate the simplex method and genetic algorithm to produce a hybrid intelligent algorithm for solving the uncertain facility location-allocation model (3). The hybrid intelligent algorithm is described in detail as the following procedures:

Step 1. From the potential region $\{(\boldsymbol{x}, \boldsymbol{y}) | g_i(x_i, y_i) \le 0, i = 1, 2, \ldots, n\}$, initialize $pop\ size$ chromosomes $V^k = (\boldsymbol{x}^k, \boldsymbol{y}^k) = (x_1^k, x_2^k, \ldots, x_n^k, y_1^k, y_2^k, \ldots, y_n^k)$, $k = 1, 2, \ldots, pop\ size$, which denote the locations of all the facilities.

Step 2. Calculate the objective values U^k for all chromosomes V^k, $k = 1, 2, \ldots, pop\ size$, where the simplex method is used to solve (5) to get the optimal cost $C(\boldsymbol{x}^k, \boldsymbol{y}^k, \Phi_1^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))$.

Step 3. Compute the fitness of all chromosomes V^k, $k = 1, 2, \ldots, pop\ size$. The rank-based evaluation function is defined as

$$Eval(V^k) = \beta(1 - \beta)^{k-1}, \ k = 1, 2, \ldots, pop\ size$$

where the chromosomes $V^1, V^2, \ldots, V^{pop\ size}$ are assumed to have been rearranged from good to bad according to their objective values U^k and $\beta \in (0, 1)$ is a parameter in the genetic system.

Step 4. Select the chromosomes for a new population. The selection process is based on spinning the roulette wheel characterized by the fitness of all chromosomes for $pop\ size$ times, and each time we select a single chromosome. Thus we update the $pop\ size$ chromosomes V^k, $k = 1, 2, \ldots, pop\ size$.

Step 5. Update the chromosomes V^k, $k = 1, 2, \ldots, pop\ size$ by crossover operation. We define a parameter P_c of a genetic system as the probability of crossover. This probability gives us the expected number $P_c \cdot pop\ size$ chromosomes undergoing the crossover operation.

Step 6. Update the chromosomes V^k, $k = 1, 2, \ldots, pop\ size$ by mutation operation. The parameter P_m is the probability of mutation, which gives us the expected number of $P_m \cdot pop\ size$ chromosomes undergoing the mutation operations.

Step 7. Repeat the second to the sixth steps for a given number of cycles.

Step 8. Report the best chromosome $V^* = (\boldsymbol{x}^*, \boldsymbol{y}^*)$ as the optimal locations.

5 A numerical example

Consider a company that wishes to locate four new facilities in a square region $[0, 100] \times [0, 100]$, whose capacities are 100, 110, 120 and 130, respectively. Assume that there are 20 customers whose demands are all zigzag uncertain variables. The location (a_j, b_j) and the uncertain demand ξ_j of the customer j are given in Table 1 for $j = 1, 2, \ldots, 20$.

Table 1 Location and uncertain demand

Customer j	Location	Demand	Customer j	Location	Demand
1	(28, 42)	$\mathcal{Z}(14, 15, 17)$	11	(14, 78)	$\mathcal{Z}(13, 15, 17)$
2	(18, 50)	$\mathcal{Z}(13, 14, 18)$	12	(90, 36)	$\mathcal{Z}(11, 14, 17)$
3	(74, 34)	$\mathcal{Z}(12, 14, 16)$	13	(78, 20)	$\mathcal{Z}(13, 15, 19)$
4	(74, 6)	$\mathcal{Z}(17, 18, 20)$	14	(24, 52)	$\mathcal{Z}(11, 13, 16)$
5	(70, 18)	$\mathcal{Z}(21, 23, 26)$	15	(54, 6)	$\mathcal{Z}(20, 24, 26)$
6	(72, 98)	$\mathcal{Z}(24, 26, 28)$	16	(62, 60)	$\mathcal{Z}(16, 18, 23)$
7	(60, 50)	$\mathcal{Z}(13, 15, 16)$	17	(98, 14)	$\mathcal{Z}(18, 19, 22)$
8	(36, 40)	$\mathcal{Z}(12, 14, 17)$	18	(36, 58)	$\mathcal{Z}(13, 14, 17)$
9	(12, 4)	$\mathcal{Z}(13, 15, 17)$	19	(38, 88)	$\mathcal{Z}(16, 17, 20)$
10	(18, 20)	$\mathcal{Z}(22, 24, 26)$	20	(32, 54)	$\mathcal{Z}(19, 22, 25)$

It follows from the α-cost minimization model (3) that the company's problem can be formulated as

$$\begin{cases} \min\limits_{x,y,z} \sum\limits_{i=1}^{4} \sum\limits_{j=1}^{20} z_{ij}\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \\ \text{subject to :} \\ \qquad \sum\limits_{i=1}^{4} z_{ij} \geq \Phi_j^{-1}(\alpha), \quad j = 1, 2, \ldots, 20 \\ \qquad \sum\limits_{j=1}^{20} z_{1j} \leq 100 \\ \qquad \sum\limits_{j=1}^{20} z_{2j} \leq 110 \\ \qquad \sum\limits_{j=1}^{20} z_{3j} \leq 120 \\ \qquad \sum\limits_{j=1}^{20} z_{4j} \leq 130 \\ \qquad z_{ij} \geq 0, \quad i = 1, 2, 3, 4, \ j = 1, 2, \ldots, 20 \\ \qquad 0 \leq x_i \leq 100, \quad i = 1, 2, 3, 4 \\ \qquad 0 \leq y_i \leq 100, \quad i = 1, 2, 3, 4. \end{cases} \tag{6}$$

In order to solve the model (6), the hybrid intelligent algorithm was run with 1,000 generations. The results for different α are given in Table 2, which shows that the minimal α-cost increases as the confidence level α increases.

When the confidence level α is taken as 0.9, the locations of customers and the optimal locations of the facilities are shown in Fig. 1, in which the points represent the locations of the customers and the diamonds represent the optimal locations of the facilities.

Table 2 The results of the example with different α

Confidence level α	Locations of facilities	Minimal α-cost
0.6	(78, 20), (62, 61), (21, 17), (29, 53)	5,781
0.7	(78, 20), (62, 60), (21, 16), (31, 53)	5,939
0.8	(78, 20), (61, 65), (21, 16), (29, 53)	6,010
0.9	(78, 20), (62, 60), (21, 17), (29, 53)	6,251

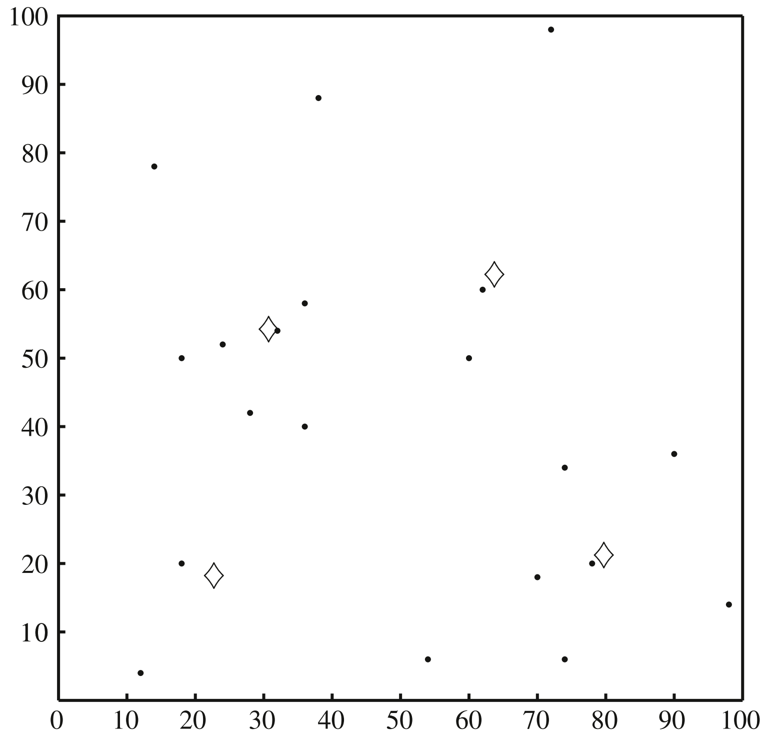

Fig. 1 Locations of customers and facilities, where *cdot* denote a customer and *diamond* denote a facility

6 Conclusion

This paper mainly proposed an α-cost minimization model for facility location-allocation problem with uncertain demands. An equivalent crisp model was obtained, based on which a hybrid intelligent algorithm integrating simplex method and genetic algorithm was designed. A numerical experiment was given to illustrate the hybrid intelligent algorithm.

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Research Article

Data Envelopment Analysis with Uncertain Inputs and Outputs

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Data envelopment analysis (DEA), as a useful management and decision tool, has been widely used since it was first invented by Charnes et al. in 1978. On the one hand, the DEA models need accurate inputs and outputs data. On the other hand, in many situations, inputs and outputs are volatile and complex so that they are difficult to measure in an accurate way. The conflict leads to the researches of uncertain DEA models. This paper will consider DEA in uncertain environment, thus producing a new model based on uncertain measure. Due to the complexity of the new uncertain DEA model, an equivalent deterministic model is presented. Finally, a numerical example is presented to illustrate the effectiveness of the uncertain DEA model.

1. Introduction

Data envelopment analysis is a mathematical programming technique that measures the relative efficiency of decision making units with multiple inputs and outputs, which was initialized by Charnes et al. [1]. This was followed by variety of theory research work, including Banker et al. [2], Charnes et al. [3], Petersen [4], and Tone [5]. More DEA papers can refer to Seiford [6] in which 500 references are documented.

The original DEA models assume that inputs and outputs are measured by exact values. However, in many situations, such as in a manufacturing system, a production process, or a service system, inputs and outputs are volatile and complex so that they are difficult to measure in an accurate way. Thus many researchers tried to model DEA with various uncertain theories. Probability theory is the earliest theory which was used to establish the stochastic DEA models. Sengupta [7] generalized the stochastic DEA model using the expected value. Banker [8] incorporated statistical elements into DEA, thus developing a statistical method. Many papers [9–13] have employed the chance-constrained programming to DEA in order to accommodate stochastic variations in data. Fuzzy theory is another theory which was used to deal with the uncertainty in DEA. As one of the DEA initiators, Cooper et al. [14–16] introduced how to deal with imprecise data such as bounded data, ordinal data, and ratio bounded data in DEA. Kao and Liu [17] developed a method to find the membership functions of the fuzzy efficiency scores when some inputs or inputs are fuzzy numbers. Entani et al. [18] proposed a DEA model with an interval efficiency by the pessimistic and the optimistic values. Many researchers have introduced possibility measure [19] into DEA [20, 21].

A lot of surveys showed that human uncertainty does not behave like fuzziness. For example, we say “the input is about 10.” Generally, we employ fuzzy variable to describe the concept of “about 10;” then there exists a membership function, such as a triangular one (9, 10, 11). Based on this membership function, we can obtain that “the input is exactly 10” with possibility measure 1. On the other hand, the opposite event of “not exactly 10” has the same possibility measure. The conclusion that “not 10” and “exactly 10” have the same possibility measure is not appropriate. This inspired Liu [22] to found an uncertainty theory which has become a branch of axiomatic mathematics for modeling human uncertainty. This paper will apply the uncertainty theory to DEA to deal with human uncertainty, thus producing some uncertain DEA models.

In this paper, we will assume the inputs and outputs are uncertain variables and propose some uncertain DEA models. The rest of this paper is organized as follows. Section 2 will introduce some basic concepts and properties about uncertain variables. Then an uncertain DEA model as well as its equivalent crisp model will be presented in Section 3. Finally, a numerical example will be given to illustrate the uncertain DEA model in Section 4.

2. Preliminaries

Uncertainty theory was founded by Liu [22] in 2007 and refined by Liu [23] in 2010. As extensions of uncertainty theory, uncertain process, and uncertain differential equations [24], uncertain calculus [25] were proposed. Besides, uncertain programming was first proposed by Liu [26] in 2009, which wants to deal with the optimal problems involving uncertain variable. This work was followed by an uncertain multiobjective programming, an uncertain goal programming [27], and an uncertain multilevel programming [28]. Since that, uncertainty theory was used to solve variety of real optimal problems, including finance [29–31], reliability analysis [32, 33], graph [34, 35], and train scheduling [36, 37]. In this section, we will state some basic concepts and results on uncertain variables. These results are crucial for the remainder of this paper.

Let Γ be a nonempty set, and let Ł be a σ-algebra over Γ. Each element $\Lambda \in$ Ł is assigned a number $M\{\Lambda\} \in [0,1]$. In order to ensure that the number $M\{\Lambda\}$ has certain mathematical properties, Liu [22] presented the four axioms.

Axiom 1. $M\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3. For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}. \quad (1)$$

Axiom 4. Let $(\Gamma_k, Ł_k, M_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then the product uncertain measure M is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}, \quad (2)$$

where Λ_k are arbitrarily chosen events from $Ł_k$ for $k = 1, 2, \ldots$, respectively.

If the set function M satisfies the first three axioms, it is called an uncertain measure.

Definition 1 (see Liu [22]). Let Γ be a nonempty set, let Ł be a σ-algebra over Γ, and let M be an uncertain measure. Then the triplet $(\Gamma, Ł, M)$ is called an uncertainty space.

Definition 2 (see Liu [22]). An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, Ł, M)$ to the set of real numbers; that is, for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \quad (3)$$

is an event.

Definition 3 (see Liu [22]). The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = M\{\xi \le x\} \quad (4)$$

for any real number x.

Example 4. The linear uncertain variable $\xi \sim \mathscr{L}(a,b)$ has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a, \\ \dfrac{(x-a)}{(b-a)}, & \text{if } a \le x \le b, \\ 1, & \text{if } x \ge b. \end{cases} \quad (5)$$

Example 5. An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a, \\ \dfrac{(x-a)}{2(b-a)}, & \text{if } a \le x \le b, \\ \dfrac{(x+c-2b)}{2(c-b)}, & \text{if } b \le x \le c, \\ 1, & \text{if } x \ge c \end{cases} \quad (6)$$

denoted by $\mathscr{Z}(a,b,c)$, where a, b, c are real numbers with $a < b < c$.

Definition 6 (see Liu [25]). The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$M\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\} \quad (7)$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Definition 7 (see Liu [23]). An uncertainty distribution Φ of an uncertain variable ξ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$. In this case, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Example 8. The inverse uncertainty distribution of a zigzag uncertain variable $\mathscr{Z}(a,b,c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha \le 0.5, \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha > 0.5. \end{cases} \quad (8)$$

Theorem 9 (see Liu [23]). *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions*

$\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \quad (9)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right). \quad (10)$$

Example 10. Let ξ be an uncertain variable with regular uncertainty distribution Φ. Since $f(x) = ax + b$ is a strictly increasing function for any constants $a > 0$ and b, the inverse uncertainty distribution of $a\xi + b$ is

$$\Psi^{-1}(\alpha) = a\Phi_1^{-1}(\alpha) + b. \quad (11)$$

Example 11. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Since

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n \quad (12)$$

is a strictly increasing function, the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \quad (13)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha). \quad (14)$$

Theorem 12 (see Liu [23]). *Assume the constraint function $g(x, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$. If $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, then the chance constraint*

$$M\{g(x, \xi_1, \xi_2, \ldots, \xi_n) \le 0\} \ge \alpha \quad (15)$$

holds if and only if

$$g\left(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right) \le 0. \quad (16)$$

3. DEA Model

In many situations, inputs and outputs are volatile and complex so that they are difficult to measure in an accurate way. This inspired many researchers to apply probability to DEA. As we know, probability or statistics needs a large amount of historical data. In the vast majority of real cases, the sample size is too small (even no sample) to estimate a probability distribution. Then we have to invite some domain experts to evaluate their degree of belief that each event will occur. This section will give some researches to empirical uncertain DEA using the theory introduced in Section 2. The new symbols and notations are given as follows:

DMU_i: the ith DMU, $i = 1, 2, \ldots, n$;

DMU_0: the target DMU;

$\tilde{x}_k = (\tilde{x}_{k1}, \tilde{x}_{k2}, \ldots, \tilde{x}_{kp})$: the uncertain inputs vector of DMU_k, $k = 1, 2, \ldots, n$;

$\Phi_{ki}(x)$: the uncertainty distribution of $\tilde{x}_{ki}$, $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, p$;

$\tilde{x}_0 = (\tilde{x}_{01}, \tilde{x}_{02}, \ldots, \tilde{x}_{0p})$: the inputs vector of the target DMU_0;

$\Phi_{0i}(x)$: the uncertainty distribution of $\tilde{x}_{0i}$, $i = 1, 2, \ldots, p$;

$\tilde{y}_k = (\tilde{y}_{k1}, \tilde{y}_{k2}, \ldots, \tilde{y}_{kq})$: the uncertain outputs vector of DMU_k, $k = 1, 2, \ldots, n$;

$\Psi_{kj}(x)$: the uncertainty distribution of $\tilde{y}_{kj}$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, q$;

$\tilde{y}_0 = (\tilde{y}_{01}, \tilde{y}_{02}, \ldots, \tilde{y}_{0q})$: the outputs vector of the target DMU_0;

$\Psi_{0j}(x)$: the uncertainty distribution of $\tilde{y}_{0j}$, $j = 1, 2, \ldots, q$.

3.1. Uncertainty Distributions of Inputs and Outputs. Liu and Ha [38] proposed a questionnaire survey for collecting expert's experimental data. It is based on expert's experimental data rather than historical data. The starting point is to invite one expert who is asked to complete a questionnaire about the meaning of an uncertain input (output) ξ like "How many is the input (output)."

We first ask the domain expert to choose a possible value x that the uncertain input ξ may take and then quiz him,

"How likely is ξ less than or equal to x?"

Denote the expert's belief degree by α. An expert's experimental data (x, α) is thus acquired from the domain expert.

Repeating the above process, we can obtain the following expert's experimental data:

$$(x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n) \quad (17)$$

that meet the following consistence condition (perhaps after a rearrangement):

$$x_1 < x_2 < \cdots < x_n, \qquad 0 \le \alpha_1 \le \alpha_2 \le \cdots \alpha_n \le 1. \quad (18)$$

Based on those expert's experimental data, Liu and Ha [38] suggested an empirical uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le x_1, \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n, \\ 1, & \text{if } x > x_n. \end{cases} \quad (19)$$

Assume there are m domain experts and each produces an uncertainty distribution. Then we may get m uncertainty distributions $\Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x)$. The Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. Wang et al. [39] recast the Delphi method as a process to determine the uncertainty distribution. The main steps are listed as follows.

Step 1. The m domain experts provide their expert's experimental data,

$$\left(x_{ij},\alpha_{ij}\right), \quad j=1,2,\dots,n_i,\ i=1,2,\dots,m. \tag{20}$$

Step 2. Use the ith expert's experimental data $(x_{i1},\alpha_{i1}),(x_{i2},\alpha_{i2}),\dots,(x_{in_i},\alpha_{in_i})$ to generate the ith expert's uncertainty distribution Φ_i.

Step 3. Compute $\Phi(x)=w_1\Phi_1(x)+w_2\Phi_2(x)+\cdots+w_m\Phi_m(x)$, where $w_1,w_2,\dots,w_m$ are convex combination coefficients.

Step 4. If $|\alpha_{ij}-\Phi(x_{ij})|$ are less than a given level $\varepsilon>0$, then go to Step 5. Otherwise, the ith expert receives the summary (Φ and reasons) and then provides a set of revised expert's experimental data. Go to Step 2.

Step 5. The last Φ is the uncertainty distribution of the input (output).

3.2. Uncertain DEA Model. Similar to traditional DEA model [3], the objective of the uncertain DEA model is to maximize the total slacks in inputs and outputs subject to the constraints. Then the uncertain DEA model can be given as follows:

$$\begin{aligned}
\max\quad & \sum_{i=1}^{p}s_i^- + \sum_{j=1}^{q}s_j^+ \\
\text{subject to:}\quad & M\left\{\sum_{k=1}^{n}\tilde{x}_{ki}\lambda_k \le \tilde{x}_{0i}-s_i^-\right\}\ge\alpha,\quad i=1,2,\dots,p,\\
& M\left\{\sum_{k=1}^{n}\tilde{y}_{kj}\lambda_k \ge \tilde{y}_{0j}+s_j^+\right\}\ge\alpha,\quad j=1,2\dots,q,\\
& \sum_{k=1}^{n}\lambda_k=1,\\
& \lambda_k\ge 0,\quad k=1,2,\dots,n,\\
& s_i^-\ge 0,\quad i=1,2\dots,p,\\
& s_j^+\ge 0,\quad j=1,2,\dots,q.
\end{aligned} \tag{21}$$

Definition 13 (α-efficiency). DMU_0 is α-efficient if s_i^{-*} and s_j^{+*} are zero for $i=1,2,\dots,p$ and $j=1,\dots,q$, where s_i^{-*} and s_j^{+*} are optimal solutions of (21).

Since the uncertain measure is involved, this definition is different from traditional efficiency definition. For instance, as determined by the choice of α, there is a risk that DMU_0 will not be efficient even when the condition of Definition 13 is satisfied.

Since $j=0$ is one of the DMU_j, we can always get a solution with $\lambda_0=1$, $\lambda_j=0$ ($j\ne 0$), and all slacks zero. Thus this uncertain DEA model has feasible solution and the optimal value $s_i^{-*}=s_j^{+*}=0$ for all i, j.

3.3. Deterministic Model. Model (21) is an uncertain programming model, which is too complex to compute directly. This section will give its equivalent crisp model to simplify the computation process.

Theorem 14. *Assume that $\tilde{x}_{1i},\tilde{x}_{2i},\dots,\tilde{x}_{ni}$ are independent uncertain inputs with uncertainty distribution $\Phi_{1i},\Phi_{2i},\dots,\Phi_{ni}$ for each i, $i=1,2,\dots,p$, and $\tilde{y}_{1i},\tilde{y}_{2i},\dots,\tilde{y}_{ni}$ are independent uncertain outputs with uncertainty distribution $\Psi_{1j},\Psi_{2j},\dots,\Psi_{nj}$ for each j, $j=1,2,\dots,q$. Then*

$$\begin{aligned}
& M\left\{\sum_{k=1}^{n}\tilde{x}_{ki}\lambda_k \le \tilde{x}_{0i}-s_i^-\right\}\ge\alpha,\quad i=1,2,\dots,p,\\
& M\left\{\sum_{k=1}^{n}\tilde{y}_{kj}\lambda_k \ge \tilde{y}_{0j}+s_j^+\right\}\ge\alpha,\quad j=1,2\dots,q
\end{aligned} \tag{22}$$

holds if and only if

$$\begin{aligned}
& \sum_{k=1,k\ne 0}^{n}\lambda_k\Phi_{ki}^{-1}(\alpha)+\lambda_0\Phi_{0i}^{-1}(1-\alpha)\le\Phi_{0i}^{-1}(1-\alpha)-s_i^-,\\
& \qquad i=1,2,\dots,p,\\
& \sum_{k=1,k\ne 0}^{n}\lambda_k\Psi_{kj}^{-1}(1-\alpha)+\lambda_0\Psi_{0j}^{-1}(\alpha)\ge\Psi_{0j}^{-1}(\alpha)+s_j^-,\\
& \qquad j=1,2,\dots,q.
\end{aligned} \tag{23}$$

Proof. Without loss of generality, let $i=1$ and $x_0=x_1$; then we will consider the equation

$$M\left\{\sum_{k=1}^{n}\tilde{x}_{k1}\lambda_k \le \tilde{x}_{11}-s_i^-\right\}\ge\alpha. \tag{24}$$

Rewrite (24) as

$$M\left\{\sum_{k=2}^{n}\tilde{x}_{k1}\lambda_k-\left(1-\lambda_1\right)\tilde{x}_{11}\le -s_i^-\right\}\ge\alpha. \tag{25}$$

Since $-(1-\lambda_1)\tilde{x}_{11}$ is an uncertain variable which is decreasing with respect to $\tilde{x}_{11}$, its inverse uncertainty distribution is

$$\Upsilon_{11}^{-1}(\alpha)=-\left(1-\lambda_1\right)\Phi_{11}^{-1}(1-\alpha),\quad 0<\alpha<1. \tag{26}$$

For each $2\le k\le n$, $\tilde{x}_{k1}\lambda_k$ is an uncertain variable whose inverse uncertainty distribution is

$$\Upsilon_{k1}^{-1}(\alpha)=\lambda_k\Phi_{k1}^{-1}(\alpha),\quad 0<\alpha<1. \tag{27}$$

Table 1: DMUs with two uncertain inputs and two uncertain outputs.

DMU_i	1	2	3	4	5
Input 1	$\mathcal{Z}(3.5, 4.0, 4.5)$	$\mathcal{Z}(2.9, 2.9, 2.9)$	$\mathcal{Z}(4.4, 4.9, 5.4)$	$\mathcal{Z}(3.4, 4.1, 4.8)$	$\mathcal{Z}(5.9, 6.5, 7.1)$
Input 2	$\mathcal{Z}(2.9, 3.1, 3.3)$	$\mathcal{Z}(1.4, 1.5, 1.6)$	$\mathcal{Z}(3.2, 3.6, 4.0)$	$\mathcal{Z}(2.1, 2.3, 2.5)$	$\mathcal{Z}(3.6, 4.1, 4.6)$
Output 1	$\mathcal{Z}(2.4, 2.6, 2.8)$	$\mathcal{Z}(2.2, 2.2, 2.2)$	$\mathcal{Z}(2.7, 3.2, 3.7)$	$\mathcal{Z}(2.5, 2.9, 3.3)$	$\mathcal{Z}(4.4, 5.1, 5.8)$
Output 2	$\mathcal{Z}(3.8, 4.1, 4.4)$	$\mathcal{Z}(3.3, 3.5, 3.7)$	$\mathcal{Z}(4.3, 5.1, 5.9)$	$\mathcal{Z}(5.5, 5.7, 5.9)$	$\mathcal{Z}(6.5, 7.4, 8.3)$

Table 2: Results of evaluating the DMUs with $\alpha = 0.6$.

DMUs	$(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*, \lambda_5^*)$	$\sum_{i=1}^{p} s_i^{-*} + \sum_{j=1}^{q} s_j^{+*}$	The result of evaluating
DMU_1	(0, 0.25, 0, 0.75, 0)	1.89	Inefficiency
DMU_2	(0, 1, 0, 0, 0)	0	Efficiency
DMU_3	(0, 0, 0, 0.78, 0.22)	1.54	Inefficiency
DMU_4	(0, 0, 0, 1, 0)	0	Efficiency
DMU_5	(0, 0, 0, 0, 1)	0	Efficiency

It follows from the operational law that the inverse uncertainty distribution of the sum $\sum_{k=2}^{n} \tilde{x}_{k1}\lambda_k - (1-\lambda_1)\tilde{x}_{11}$ is

$$\Upsilon^{-1}(\alpha) = \sum_{k=1}^{n} \Upsilon_{21}^{-1}(\alpha) = \sum_{k=2}^{n} \lambda_k \Phi_{k1}^{-1}(\alpha) - (1-\lambda_1)\Phi_{11}^{-1}(1-\alpha), \quad 0 < \alpha < 1. \tag{28}$$

From which we may derive the result immediately for $i = 1$ and $x_0 = x_1$. Similarly, we can get other results.

Following Theorem 14, the uncertain DEA model can be converted to the crisp model as follows:

$$\begin{aligned} \max \quad & \sum_{i=1}^{p} s_i^- + \sum_{j=1}^{q} s_j^+ \\ \text{subject to:} \quad & \sum_{k=1,k\neq 0}^{n} \lambda_k \Phi_{ki}^{-1}(\alpha) + \lambda_0 \Phi_{0i}^{-1}(1-\alpha) \\ & \qquad \leq \Phi_{0i}^{-1}(1-\alpha) - s_i^-, \quad i = 1, 2, \ldots, p, \\ & \sum_{k=1,k\neq 0}^{n} \lambda_k \Psi_{kj}^{-1}(1-\alpha) + \lambda_0 \Psi_{0j}^{-1}(\alpha) \\ & \qquad \geq \Psi_{0j}^{-1}(\alpha) + s_j^+, \quad j = 1, 2, \ldots, q, \\ & \sum_{k=1}^{n} \lambda_k = 1, \\ & \lambda_k \geq 0, \quad k = 1, 2, \ldots, n, \\ & s_i^- \geq 0, \quad i = 1, 2 \ldots, p, \\ & s_j^+ \geq 0, \quad j = 1, 2, \ldots, q \end{aligned} \tag{29}$$

which is a linear programming model. Thus it can be easily solved by many traditional methods. □

4. A Numerical Example

This example wants to illustrate the uncertain DEA model. For simplicity, we will only consider five DMUs with two inputs and two outputs which are all zigzag uncertain variables denoted by $\mathcal{Z}(a, b, c)$. Table 1 gives the information of the DMUs.

For illustration, let DMU_1 be the target DMU; then the uncertain DEA model (29) can be written as

$$\begin{aligned} \max \quad & s_1^- + s_2^- + s_1^+ + s_2^+ \\ \text{subject to:} \quad & \sum_{k=2}^{5} \lambda_k \Phi_{k1}^{-1}(\alpha) + \lambda_1 \Phi_{11}^{-1}(1-\alpha) \\ & \qquad \leq \Phi_{11}^{-1}(1-\alpha) - s_1^-, \\ & \sum_{k=2}^{5} \lambda_k \Phi_{k2}^{-1}(\alpha) + \lambda_1 \Phi_{12}^{-1}(1-\alpha) \\ & \qquad \leq \Phi_{12}^{-1}(1-\alpha) - s_2^-, \\ & \sum_{k=2}^{5} \lambda_k \Psi_{k1}^{-1}(1-\alpha) + \lambda_1 \Psi_{11}^{-1}(\alpha) \geq \Psi_{11}^{-1}(\alpha) + s_1^+, \\ & \sum_{k=2}^{5} \lambda_k \Psi_{k2}^{-1}(1-\alpha) + \lambda_1 \Psi_{12}^{-1}(\alpha) \geq \Psi_{12}^{-1}(\alpha) + s_2^+, \\ & \sum_{k=1}^{5} \lambda_k = 1, \\ & \lambda_k \geq 0, \quad k = 1, 2, \ldots, 5, \end{aligned}$$

Table 3: Results of evaluating the DMUs with different confidence level α.

α	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
0.5	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.6	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.7	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.8	Inefficiency	Efficiency	Efficiency	Efficiency	Efficiency
0.9	Efficiency	Efficiency	Efficiency	Efficiency	Efficiency

$$\begin{aligned} & s_1^- \geq 0, \\ & s_2^- \geq 0, \\ & s_1^+ \geq 0, \\ & s_2^+ \geq 0. \end{aligned} \tag{30}$$

Table 2 shows the results of evaluating DMUs with confidence level $\alpha = 0.6$. The results can be interpreted in the following way: DMU_1 and DMU_3 are inefficient, whereas DMU_2, DMU_4, and DMU_5 are efficient. Moreover, DMU_3 is more efficient than DMU_1 from the total slacks $\sum_{i=1}^{p} s_i^{-*} + \sum_{j=1}^{q} s_j^{+*}$, since they are both inefficient.

Uncertain efficiencies obtained from model (30) for different confidence levels α are shown in Table 3. DMU_1 is inefficient at all confidence levels, whereas DMU_2, DMU_4, and DMU_5 are always efficient at all levels. It can be seen that the number of the efficient DMUs is affected by the confidence level α. The higher the confidence level α is, the bigger the number of efficient DMUs is. This phenomena indicate that uncertain DEA is more complex than the traditional DEA because of the inherent uncertainty contained in inputs and outputs.

5. Conclusion

Due to its widely practical used background, data envelopment analysis (DEA) has become a pop area of research. Since the data cannot be precisely measured in some practical cases, many papers have been published when the inputs and outputs are uncertain. This paper has given some researches to uncertain DEA model. A new DEA model as well as its equivalent deterministic model was presented. For illustration, a numerical example was designed.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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(s, S) POLICY FOR UNCERTAIN SINGLE PERIOD INVENTORY PROBLEM

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The traditional single period inventory problem assumes that the market demand is a random variable. However, as an empirical or subjective estimation, market demand is better to be regarded as an uncertain variable. This paper is concerning with single period inventory problem under two main assumptions that (i) the market demand is an uncertain variable and (ii) a setup cost and an initial stock exist. Under the framework of uncertainty theory, the optimal inventory policy for uncertain single period inventory problem with an initial stock and a setup cost is derived, which is of (s, S) type. Also, some expansions are obtained.

Keywords: Inventory; (s, S) policy; uncertainty theory.

1. Introduction

The single period inventory problem is to find an optimal inventory policy which maximizes the expected profit, or equivalently minimizes the expected cost. The single period inventory problem has two significant assumptions: (i) the market

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demand during the period is non-deterministic, and (ii) items can only be ordered or produced in batch at the beginning of the period. The traditional single period inventory problem also assumes that if any inventory remains at the end of the period, the excess inventory is sold by a discount or disposed off simply; on the other hand, if market demand exceeds the inventory level, some profit is lost. These assumptions are proper in most cases, such as in fashing, sporting and service industries. As a result, after been introduced by Hadley and Whitin,[7] a lot of extensions to single period inventory problem have been developed, such as Bassok *et al.*,[1] Ehrhardt,[4] Fu *et al.*,[5] Lau *et al.*,[10] Sana,[20] etc.

At first, the non-deterministic demand of single period inventory problem was regarded as a random variable. So, the above research work employed probability to deal with the non-deterministic demand. In 1996, Petrović *et al.*[18] used fuzzy set to describe the demand, which opened the door of introducing fuzzy theory into single period inventory problem. Some extension fuzzy models were investigated by Dutta *et al.*,[3] Ishill *et al.*,[8] Ji *et al.*,[9] Li *et al.*,[11] etc.

With the development of theories on non-deterministic phenomenon, it is found that some non-deterministic phenomenon cannot be described by randomness or fuzziness, such as the empirical estimate of demand in a future period. In order to describe this type of non-deterministic phenomenon, uncertainty theory was proposed by Liu[12] in 2007 and refined by Liu[16] in 2010. In 2009, Qin and Kar[19] introduced uncertainty theory into single period inventory problem, and they regarded the market demand as an uncertain variable. In their paper, they derived the optimal order quantity which maximizes the expected profit.

However, Qin and Kar's model is a simple one with neither a setup cost nor an initial stock. In real life, either the setup cost or initial stock has to be taken into account. Then, the inventory policy must be changed correspondingly. This paper expands Qin and Kar's model to a more complex one, which concerns with an initial stock and a setup cost. Still, the market demand in this paper is regarded as an uncertain variable. The main contribution of this paper is to derive the optimal policy of single period inventory problem with an initial stock and a setup cost, which is of (s, S) type.

The rest of the paper is organized as follows. In Sec. 2, uncertainty theory is introduced in several paragraphs, and some basic concepts and properties of uncertainty theory are presented. In Sec. 3, single period inventory problem with a setup cost and an initial stock level will be described in detail. In Sec. 4, (s, S) policy is derived as the optimal inventory policy for single period inventory problem described in Sec. 3. Section 5 concludes this paper with a brief summary.

2. Preliminary of Uncertainty Theory

In the past, when constructing mathematical models, the empirical or subjective estimation of non-deterministic information, such as "about 100 kg", "approximately 39°C", "big size" and "young", is described by random variable or fuzzy variable. However, a lot of surveys showed that it's not suitable. For example, we say "the distance between Beijing and Shanghai is about 1300 km". Obviously, "about 1300 km" is not a random variable, since it is a constant which we do not know exactly. Can it be described by fuzzy variable? The answer is no. If we employ fuzzy variable to describe the concept of "about 1300 km", then there exists a membership function, such as a triangular one (1200, 1300, 1400). Based on this membership function, possibility theory will conclude: (i) the distance between Beijing and Shanghai is "exactly 1300 km" with belief degree 1, and (ii) the distance between Beijing and Shanghai is "not 1300 km" with belief degree 1. It is a paradox. In uncertainty theory, this paradox will not happen.

Coming from the judgement of manager, the market demand of next business period is also a non-deterministic information like "about 1300 km". In order to suitably deal with these non-deterministic information, introducing uncertainty theory to describe market demand is necessary.

Founded in 2007, uncertainty theory is a new branch of mathematics. However, theory and practice have shown that uncertainty theory is an efficient tool to deal with some non-deterministic information, such as expert data and subjective estimate, which appears in many optimization problems. In theoretical aspect, in 2008, Liu[13] first introduced uncertain process, a sequence of uncertain variables indexed by time or space. Later, uncertain calculus was proposed by Liu[15] in 2009, and Chen and Liu[2] proved the existence and uniqueness theorem for uncertain differential equation. Nearly at the same time, uncertain set theory was proposed by Liu[17] in 2010 as a generalization of uncertainty theory to the domain of uncertain sets. In practical aspect, Liu[14] built the framework of uncertain programming, which was soon applied to machine scheduling problem, vehicle routing problem and project scheduling problem. Through the work of Liu[17] and Gao *et al.*,[6] uncertain inference is developed under uncertain set theory. In 2010, Liu[16] started the research of uncertain statistics, which gives an empirical uncertainty distribution from expert's experimental data. Meanwhile, Zhu[21] studied uncertain optimal control, and applied it into portfolio selection model. In short, uncertainty theory is researched and used more and more.

In this section, we introduce some foundational concepts and property of uncertainty theory, which will be used throughout this paper.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\} \in [0, 1]$. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu[12,16] presented the four axioms: normality, self-duality, countable subadditivity, and product measure axiom. If satisfying these four axioms, the set function $\mathcal{M}\{\Lambda\}$ is called an uncertain measure.

Definition 1. (Liu[12]) Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2. (Liu[12]) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x. For example, the zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/2(b-a), & \text{if } a \le x \le b \\ (x+c-2b)/2(c-b), & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c\,. \end{cases}$$

Definition 3. (Liu[16]) An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

Obviously, zigzag uncertain variable has a regular uncertainty distribution. If Φ is regular, uncertainty distribution Φ is continuous and strictly increasing at each point x with $0 < \Phi(x) < 1$. We usually assume that all uncertainty distribution in practical application is regular. Otherwise, a small perturbation can be imposed to obtain a regular one.

Definition 4. (Liu[12]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}dr - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}dr$$

provided that at least one of the two integrals is finite.

Example 1: The expected value of zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ is $E[\xi] = (a + 2b + c)/4$.

3. Problem Description

Generally, there are two equivalent approaches to follow when constructing the objective function for single period inventory problem. One is to maximize the expected value of total profit during the period; another is to minimize the expected value of cost. In this paper, we follow the first one. The notation and assumptions are listed as below.

3.1. *Notation*

ξ	quantity demand, which is an uncertain variable;
y	inventory level, which is a decision variable;
x	initial stock;
K	setup cost;
c	purchasing or production cost of per unit;
h	salvage value of per unit;
p	selling price of per unit;
$r(\xi, y)$	the revenue for demand ξ and inventory y;
$R(y)$	the expected revenue for inventory level y, i.e., $R(y) = E[r(\xi, y)]$;
$f(\xi, x)$	the profit for demand ξ and initial stock x;
$F(x)$	the expected profit for initial stock x, i.e., $F(x) = E[f(\xi, x)]$.

3.2. *Assumptions*

(1) K, c, h and p are constant and independent of inventory policy and market demand;
(2) Shortages are permitted, and there is no shortage cost other than loss in revenue;
(3) The salvage value h is positive, and $p > c > h > 0$;
(4) There is no budget constraint.

3.3. *Mathematical formulation*

It is assumed that the initial stock is $x \geq 0$, the setup cost is $K > 0$, and no item will be ordered during the period.

Because the setup cost is positive, any order will lead to a positive cost K. This means that ordering nothing may yield the maximum expected profit in some situation.

According to the given notation and assumptions, if order nothing, the revenue during the period is

$$r(\xi, y) = r(\xi, x) = \begin{cases} px, & \text{if } x \leq \xi \\ p\xi + h(x - \xi), & \text{if } x \geq \xi \,. \end{cases}$$

Ordering nothing means no cost happens, then the total profit is just the revenue, that is

$$f(\xi, x) = \begin{cases} px, & \text{if order nothing and } x \leq \xi \\ p\xi + h(x - \xi), & \text{if order nothing and } x \geq \xi \,. \end{cases}$$

If order up to $y > x$, the revenue during the period is

$$r(\xi, y) = \begin{cases} py, & \text{if } y \leq \xi \\ p\xi + h(y - \xi), & \text{if } y \geq \xi \,. \end{cases}$$

Abstracting the cost $(c(y-x)+K)$ from revenue, we obtain the total profit

$$f(\xi,x)=r(\xi,y)-c(y-x)-K=\begin{cases} py-c(y-x)-K, \\ \qquad\qquad \text{if order up to } y \text{ and } y\le\xi \\ p\xi+h(y-\xi)-c(y-x)-K, \\ \qquad\qquad \text{if order up to } y \text{ and } y\ge\xi\,. \end{cases}$$

Obviously, $r(\xi,y)$ and $f(\xi,x)$ are both uncertain variables. Taking the expected value of $r(\xi,y)$ and $f(\xi,x)$, we get

$$F(x)=E[f(\xi,x)]=\begin{cases} R(x), & \text{if order nothing} \\ R(y)-c(y-x)-K, & \text{if order up to } y, \end{cases}$$

where $R(t)=E[r(\xi,t)]$. Let $\Phi(t)$ be the uncertainty distribution of ξ, which is regular. Qin and Kar[19] proved

$$R(y)=E[r(\xi,y)]=py-(p-h)\int_0^y \Phi(r)dr\,.$$

Our objective is to seek optimal inventory policy, which maximizes the expected profit, that is

$$\max_{y>x}\left\{R(y)-c(y-x)-K\right\}\vee R(x)\,.$$

The following section will derive the optimal policy for this model, which is a special case of the (s,S) type.

4. The Optimal Inventory Policy

In order to find the maximum of the objective function, it is better to investigate the extremal property of function $(R(t)-ct)$ first. Taking the derivative and setting it equal to zero, we obtain

$$\frac{d(R(t)-ct)}{dt}=p-(p-h)\Phi(t)-c=0\,,$$

or

$$\Phi(t)=\frac{p-c}{p-h}\,. \tag{4}$$

It's assumed $p>c>h$, then $\frac{p-c}{p-h}\in(0,1)$. Since uncertainty distribution Φ is regular, there exists t satisfying expression (4). What is more, $\Phi(t)$ is an strictly increasing function on $\Re$. Then, there is only one root satisfying expression (4), denoted as $t=S$. It is easy to verify that $(R(t)-ct)$ is decreasing on $[S,+\infty)$ and increasing on $[0,S]$. That is, when $t=S$, $(R(t)-ct)$ reaches its maximum.

Define s as the smaller value of t satisfying $R(t)-ct=R(S)-cS-K$. Since $K>0$, we have $s\neq S$. See Fig. 1.

We will derive the optimal policy with the help of Figure 1. It breaks down into three cases.

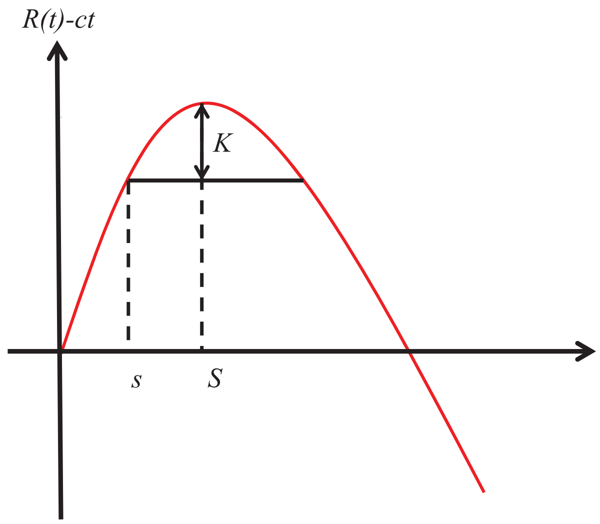

Fig. 1. Graph of $(R(t) - ct)$.

Case 1. Assume $x > S$. For any given $y \geq x$, obviously

$$R(x) - cx \geq R(y) - cy > R(y) - cy - K\,.$$

Hence, $R(x) > R(y) - c(y - x) - K$, where the right-hand side of the inequality is the expected total profit if one orders up to y, and the left-hand side is the expected total profit if one orders nothing. This indicates that if $x > S$, order nothing.

Case 2. Assume $s \leq x \leq S$. For any given $y \geq x$, obviously,

$$R(x) - cx \geq R(S) - cS - K \geq R(y) - cy - K\,.$$

Again, we get $R(x) \geq R(y) - c(y - x) - K$, which still indicates that if $s \leq x \leq S$, order nothing.

Case 3. Assume $x < s$. From Fig. 1, we obtain

$$\max_{y>x} \left\{R(y) - cy - K\right\} = R(S) - cS - K > R(x) - cx\,,$$

Hence, $R(S) - c(S - x) - K > R(x)$, that is, if $x < s$, order up to S.

This leads to an optimal policy of (s, S) type. Up to now, we can summarize:

Theorem 1. *The optimal policy for uncertain single period inventory problem with an initial stock x and a setup cost K is*

$$\begin{cases} \text{if } x < s, & \text{then order up to } S \\ \text{if } x \geq s, & \text{than order nothing,} \end{cases}$$

where the value of S satisfying $\Phi(S) = \frac{p-c}{p-h}$, and s is the smallest value of t satisfying

$$R(t) - ct = R(S) - cS - K.$$

Example 2: Assume ξ is a zigzag uncertain variable, i.e., $\xi \sim Z(9,10,12)$, $K = 40, p = 100, c = 80$ and $h = 50$. Note that S must satisfy

$$\Phi(S) = \frac{p-c}{p-h} = \frac{100-80}{100-50} = 0.4\,,$$

where $\Phi(t)$ is the uncertainty distribution of $\xi \sim Z(9,10,12)$. We can obtain the unique solution: $S = 9.8$.

Because $s < S < b = 10$, s can be obtained from

$$100s - (100-50)\int_0^s \Phi(r)dr - 80s = 100S - (100-50)\int_0^S \Phi(r)dr - 80S - 40\,,$$

where

$$\int_0^s \Phi(r)dr = \int_0^s \frac{r-9}{2}dr = \frac{s^2-18s}{4}\,,$$

$$\int_0^S \Phi(r)dr = \int_0^S \frac{r-9}{2}dr = \frac{S^2-18S}{4} = -20.08\,.$$

Then, $s = 2.84$. The optimal inventory policy is: if the initial stock level $x \geq 2.84$, order nothing; if the initial stock $x < 2.84$, order up to 9.8. □

If $K = 0$, the above single period inventory problem degenerates to one with only an initial stock x. Assume the uncertain demand ξ has uncertainty distribution $\Phi(t)$. The objective function can be expressed as

$$\max_{y>x}\left\{R(y) - c(y-x)\right\} \vee R(x),$$

where

$$R(t) = pt - (p-h)\int_0^t \Phi(r)dr, \forall t > 0\,.$$

For this problem, the optimal ordering policy is

$$\begin{cases} \text{if } x < S, & \text{then order up to } S \\ \text{if } x \geq S, & \text{then order nothing,} \end{cases}$$

where the value of S satisfying $\Phi(S) = \frac{p-c}{p-h}$.

5. Conclusion

Uncertainty theory provides a new approach to describe some non-deterministic information, such as empirical and subjective estimation. This paper employed uncertainty theory to model single period inventory problem with an initial stock and a setup cost, where the market demand was regarded as an uncertain variable. It showed that the optimal policy for the uncertain single period inventory problem with an initial stock and a setup cost is of the (s, S) type. As a degeneration model, the optimal inventory policy for single period inventory problem with only an initial stock is also given in this paper.

This paper just concerned with single period inventory problem with single product. In fact, it can be extended to the problem with multiple products, which is our next research point.

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Belief reliability: a new metrics for products' reliability

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Abstract Traditional reliability metrics are based on probability measures. However, in engineering practices, failure data are often so scarce that traditional metrics cannot be obtained. Furthermore, in many applications, premises of applying these metrics are violated frequently. Thus, this paper will give some new reliability metrics which can evaluate products' reliability with few failure data. Firstly, the new metrics are defined based on uncertainty theory and then, numerical evaluation methods for them are presented. Furthermore, a numerical algorithm based on the fault tree is developed in order to evaluate systems' reliability in the context of defined metrics. Finally, the proposed metrics and evaluation methods are illustrated with some case studies.

Keywords Uncertainty theory · Reliability · Fault tree

1 Introduction

Reliability is an important property of products, defined as the ability that a component or system will perform a required function for a given period of time under

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stated operating conditions (Ebeling 2010). Reliability metrics are defined to measure products' reliability and the process to obtain them is regarded as reliability evaluation. Traditional reliability metrics are defined on the basis of probability theory and the evaluation of them is based on statistical inferences of failure data (Meeker and Escobar 1998).

Though traditional reliability metrics have achieved great success, there are scenarios where they do not function well. Cai et al. (1991) summarized that three premises must be satisfied so that probability measures can make sense: 1. Precisely defined events; 2. Probabilistic repetitiveness in the collected data; and 3. Large sample size. However, the three premises are widely violated in numerous engineering practices. Moreover, since products' reliability continues to grow, it is often impossible to get enough failure data under restricted time and expense constraints (Meeker and Hamada 1995). Thus, challenges for traditional reliability metrics are becoming more and more severe. One way to cope with these challeges is to conduct accelerated or degradation testing for pseudo life data (Nelson 1990). Many products, however, do not fail even under accelerated conditions. Reliability evaluation of such highly reliable products requires new methods and metrics.

One possible solution might be the comprehensive evaluation. There are numerous reliability tasks in a product's life cycle, such as Design for Reliability, FMECA, Simulation Tests, etc. In fact, it is these tasks that decide the reliability of a product. If they are performed effectively, high reliability can be believed even without information from life tests. In the comprehensive evaluation, the effects of these tasks are assessed by domain experts and the reliability evaluation is conducted accordingly. However, experts' judgments will lead to human uncertainty and under these situations, probability theory might yield counterintuitive results (Liu 2012). Thus, probability-based reliability metrics are inappropriate for the comprehensive evaluation. Uncertainty theory proposed by Liu (2007) and refined by Liu (2011) is considered to be a reasonable complementation of probability theory under these settings. Therefore, this paper will develop some new reliability metrics based on uncertainty theory and discuss how to apply them to evaluate the reliability of highly reliable systems.

Uncertainty theory is a branch of axiomatic mathematics dealing with human uncertainty. Since inaugurated by Liu (2007), it has been widely used by many scholars in various areas to model human uncertainty (Chen and Liu 2010; Tan and Tang 2006), etc.. Its applications in reliability were started by Liu (2010), in which reliability index was defined as a measure of systems' reliability and some simple system reliability models were discussed, such as series and parallel models. Wang (2010) introduced Liu's definition of reliability index into structural reliability analysis.

In this paper, previous work is extended to cope with challenges in reliability evaluation of highly reliable systems. The rest of the paper is organized as follows: Sect. 2 introduces some preliminaries on uncertainty theory. In Sect. 3, the needs for reliability metrics based on uncertainty theory are discussed. Section 4 gives the definitions as well as numerical evaluation methods for the proposed metrics. In Sect. 5, evaluation of systems' reliability based on the proposed metrics is discussed and a numerical evaluation method based on fault trees is presented. Some examples are also given as an illustration to the proposed method.

2 Preliminaries

Uncertainty theory is a branch of axiomatic mathematics founded by Liu (2007) and refined by Liu (2011). Let Γ be a nonempty set, and $\mathcal{L}$ a σ -algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event. An uncertain measure is a set function $\mathcal{M}$ from $\mathcal{L}$ to [0, 1] satisfying the following axioms (Liu 2007):

Axiom 1 (*Normality Axiom*) $\mathcal{M}\{\Gamma\} = 1$ for the univeral set Γ.

Axiom 2 (*Duality Axiom*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (*Subadditivity Axiom*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. A product uncertain measure was defined by Liu (2009) in order to obtain an uncertain measure of a compound event, thus producing the fourth axiom of uncertainty theory:

Axiom 4 (*Product Axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigvee_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

An uncertain variable (Liu 2007) is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.

The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent (Liu 2009) if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} (\xi_i \in B_i)\right\} = \bigvee_{i=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

In practice, an uncertain variable is described by the uncertainty distribution, defined by Liu (2007) as $\Phi(x) = \mathcal{M}\{\xi \le x\}, \forall x \in \Re$.

An uncertainty distribution is said to be regular (Liu 2011) if its inverse function Φ^{-1} exists and is unique for each $\alpha \in (0, 1)$.

The sufficient and necessary conditions that a function is an uncertainty distribution are proved by Peng and Iwamura (2010): A function $\Phi : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.

The expected value of an uncertain variable ξ is defined by Liu (2007) as an average value of the uncertain variable in the sense of uncertain measure, i.e.,

$$E[\xi] = \int_0^{\infty} (1 - \Phi(x))\, dx - \int_{-\infty}^{0} \Phi(x) dx$$

provided that at least one of the two integrals is finite. Liu (2011) proved if the uncertainty distribution Φ was regular, then the expected value can be obtained from the inverse uncertainty distribution Φ^{-1} via

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha. \tag{1}$$

The variance of an uncertain variable ξ is defined by Liu (2007) as an indication of an uncertain variable's degree of spread. Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is $V[\xi] = E[(\xi - e)^2]$. In order to obtain variance from uncertainty distribution, Liu (2011) stipulated

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \left(1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})\right) dx \\ &= 2 \int_0^{+\infty} x\,(1 - \Phi(e + x) + \Phi(e - x))\, dx. \end{aligned} \tag{2}$$

Liu (2011) developed operation laws for uncertain variables so that the distribution of functions of uncertain variables can be achieved. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)\right). \tag{3}$$

3 Needs for new reliability metrics

As stated before, reliability evaluation of highly reliable products is a challenge for reliability engineers. Consider a product whose life expectancy is 11,000 h. Such high reliability makes it impossible to obtain failure data even under accelerated conditions. Thus, evaluation methods based on failure data are not applicable for these products.

On the other hand, there is a lot of information about products' reliability other than failure data. For example, many reliability tasks are conducted in products' life cycle. If these tasks are performed effectively, high reliability can be believed even without conducting life tests. The effects of these reliability tasks can be evaluated by domain experts and a comprehensive evaluation of reliability can be conducted accordingly.

Results from the comprehensive evaluation reflect our belief degrees of the products' reliability. However, as Liu asserts (Liu 2012), it is inappropriate to regard the belief degree as probability. Let's take a series system comprising of 30 components as an example.

Suppose all components in the system have the same life distributions

$$\mathcal{N}(11000, 950^2)\text{(in hours)}$$

and are independent from each other. The reliability of the system at 8,000 h can be easily obtained, which is,

$$R_s = \left(1 - \Phi\left(\frac{8000 - 11000}{950}\right)\right)^{30} = 0.9764. \tag{4}$$

However, in reality, the real distribution is unknown to us and has to be estimated by domain experts. Since human beings usually overweight unlikely events (Tversky and Kahneman 1986), the estimated distribution might have larger variance the real one. Assume that the estimated life distribution is

$$\mathcal{N}(11000, 2850^2)\text{ (in hours).}$$

Then the estimated reliability of the system at 8,000 h will be

$$R_s = \left(1 - \Phi\left(\frac{8000 - 11000}{2850}\right)\right)^{30} = 0.0087. \tag{5}$$

Results in Eqs. (4) and (5) are at opposite poles. This fact demonstrates that sticking to probability-based metrics in a comprehensive reliability evaluation might lead to an unacceptable result. Thus, this paper develops a set of new metrics and discusses the evaluation of system's reliability based on them.

4 The new reliability metrics

4.1 Definitions

Definition 1 (*Reliability index*) (Liu 2010) Assume a system contains uncertain variables $\xi_1, \xi_1, \ldots, \xi_n$, and there is a function R such that the system is working if and only if $R(\xi_1, \xi_1, \ldots, \xi_n) \geq 0$. Then the reliability index is

$$Reliability = \mathcal{M}\{R(\xi_1, \xi_1, \ldots, \xi_n) \geq 0\}. \tag{6}$$

Remark 1 To avoid confusions with probability-based reliability index, the reliability index in Definition 1 is regarded as Belief Reliability (R_B) in this paper. Often life is assumed to be an uncertain variable T with uncertainty distribution $\Phi(t)$, so Eq. (6) becomes

$$R_B(t) = \mathcal{M}\{T > t\} = 1 - \mathcal{M}\{T \le t\} = 1 - \Phi(t). \tag{7}$$

Remark 2 Equation (7) indicates that belief reliability is a time-varying function (often decreasing). Thus, it is regarded as belief reliability function and the processes to obtain this function are called time-dependent analyses. In some applications, the focus will be the value of $R_B(t)$ at a given period $t = t_0$. These applications are regarded as time-static analyses. Both the time-static and time-dependent analyses will be discussed in detail in Sect. 5.

Definition 2 (*Belief Reliable Life*, $BL(\alpha)$) Assume that products' life is an uncertain variable T with belief reliability function $R_B(t)$ and uncertainty distribution $\Phi(t)$. Let α be a real number from (0, 1). The belief reliable life $BL(\alpha)$ is

$$BL(\alpha) = \sup\{t | R_B(t) \ge \alpha\}. \tag{8}$$

Theorem 1 *Let* $\Phi(t)$ *be a regular uncertainty distribution with inverse uncertainty distribution* $\Phi^{-1}(\alpha)$. *Then*

$$BL(\alpha) = \Phi^{-1}(1 - \alpha). \tag{9}$$

Proof It can be easily proved since a regular uncertainty distribution $\Phi(t)$ is strictly increasing at each point t with $0 < \Phi(t) < 1$. □

Definition 3 (*Mean Time to Failure*, $MTTF_B$) Assume products' life is an uncertain variable T with belief reliability function $R_B(t)$. The mean time to failure $MTTF_B$ is defined by

$$MTTF_B = \mathrm{E}[T] = \int_0^{\infty} R_B(t)\mathrm{d}t. \tag{10}$$

Definition 4 (*Variance of Life*, VL_B) Assume that products' life is an uncertain variable T and its mean time to failure is $MTTF_B$. The variance of life VL_B is defined by

$$VL_B = \mathrm{E}[(T - MTTF_B)^2]. \tag{11}$$

Comments So far the metrics, namely $R_B(t)$, $BL(\alpha)$, $MTTF_B$ and VL_B, have been defined. Based on uncertainty theory, these metrics are intended to characterize reliability when few or none failure data can be achieved so that domain experts are relied on to evaluate reliability.

Belief reliability $R_B(t)$ is a function over time and it reflects experts' personal belief degrees of products' reliability. Belief reliable life $BL(\alpha)$ is the longest duration that a product can last with its R_B greater than a given level α. Since it is in time scale, it has more straightforward meaning than $R_B(t)$. Mean time to failure $MTTF_B$ and variance of life VL_B characterize the location and variation of life distribution, respectively.

It should be noted that since the life distribution is achieved by expert's experimental data rather than actual failure data, the defined metrics might change their values with the knowledge of experts and the information they get about products' reliability. Generally speaking, the more experienced experts are, and the more information they can get about products' reliability, the more accurate these metrics will be.

4.2 Numerical evaluation methods

In practice, it is often difficult to obtain the proposed metrics analytically and numerical evaluation methods are needed. In this section, numerical methods are provided for obtaining $MTTF_B$ and VL_B from $BL(\alpha)$, respectively. Note that all uncertainty distributions discussed in this section are supposed to be regular.

Theorem 2 *(*Numerical Evaluation of $MTTF_B$*) Let T be an uncertain variable with regular uncertainty distribution $\Phi(t)$ and belief reliable life $BL(\alpha)$. Then*

$$MTTF_B = \int_0^1 BL(\alpha)\mathrm{d}\alpha. \tag{12}$$

Proof From Theorem 1, we have

$$\int_0^1 BL(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(1-\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = MTTF_B.$$

□

Theorem 3 (Numerical Evaluation of VL_B) *Let T be an uncertain variable with regular uncertainty distribution $\Phi(t)$ and belief reliable life $BL(\alpha)$. Then*

$$VL_B = 2\left[\int_0^{R_B(MTTF_B)} (MTTF_B - BL(\alpha))\alpha \mathrm{d}BL(\alpha) + \int_{R_B(MTTF_B)}^1 (BL(\alpha) - MTTF_B)(1-\alpha)\mathrm{d}BL(\alpha)\right].$$

Proof The theorem can be proved from Eq. (2) by the method of variable transformation. □

In practice, the integrals in the two theorems are often conducted numerically by computers, which allow computationally flexible evaluations of $MTTF_B$ and VL_B.

5 Systems' belief reliability

5.1 Time-static systems

A time-static system is a system whose reliability does not change over time and can be viewed as a special case of a time-dependent system at a specified $t = t_0$. Although most real systems are time-dependent, time-static analyses can help us understand the relationship between components' failures and system's failures.

The systems to be discussed are assumed to be binary-state. Therefore their corresponding time-static systems are Boolean systems. Liu (2010) had developed evaluation methods for Boolean systems through the structural function. In engineering practices, structural functions are often obtained through fault trees and fault trees are more familiar to engineers than structural functions. A typical fault tree is illustrated in Fig. 1 and readers might refer to Ebeling (2010) for details on the fault tree. The rest of this section will discuss quantitative fault tree analyses in the context of belief reliability. Quantitative analyses of a fault tree involve two steps:

1. Enumerate all minimum cut sets;
2. Calculate the system's reliability based on the minimum cut sets.

A cut set is a set of components whose failures interrupt all connections between input and output ends and cause an entire system to fail. A minimal cut set is the smallest combination of components which will cause the system's failure if they all

Fig. 1 A typical fault tree

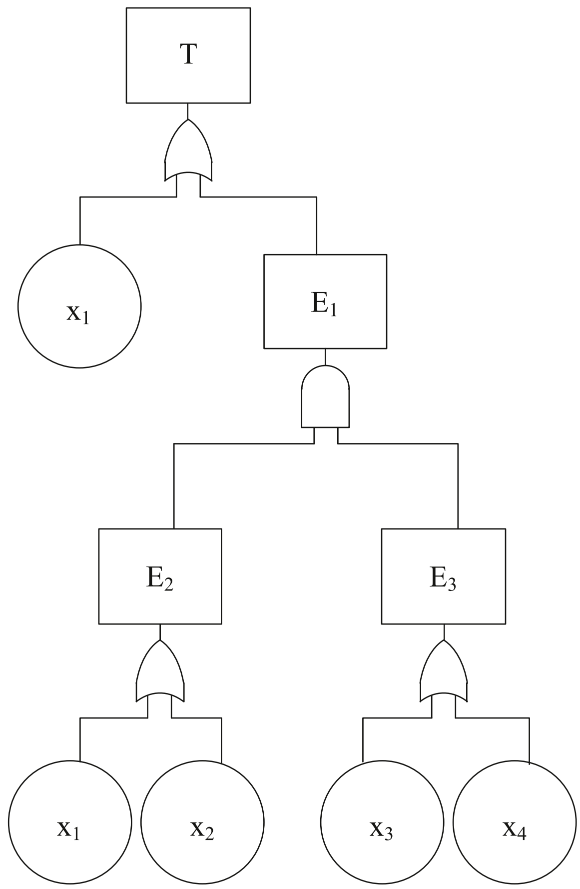


Table 1 The implementation of MOCUS

Step 1	Step 2	Step 3	Step 4
x_1	x_1	x_1	x_1
E_1	E_2, E_3	x_1, E_3	x_1, x_3
		x_2, E_3	x_1, x_4
			x_2, x_3
			x_2, x_4

fail. There are numerous algorithms for enumerating minimum cut sets from a fault tree. Among them, MOCUS by Fussell and Vesely (1972) is an efficient and widely used one and will be applied in this paper. An implementation of MOCUS to the fault tree in Fig. 1 is demonstrated blow.

- Construct an empty table.
- List all output events of an AND gate in a single row of the table, while list each of the output events in individual rows for an OR gate, as shown in Table 1. Then the last column of Table 1 lists all cut sets.
- Discard the cut sets which include other cut sets.

Since both $\{x_1, x_3\}$ and $\{x_1, x_4\}$ include $\{x_1\}$, the minimum cut sets are

$$\{x_1\}, \{x_2, x_3\}, \{x_2, x_4\}.$$

Once the minimum cut sets are decided, the belief reliability of the system can be obtained from them, as stated in Theorem 5.

Theorem 4 (Liu 2011) *Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent Boolean uncertain variables, i.e., $\xi_i = 1$ with uncertainty measure a_i for $i = 1, 2, \ldots, n$. Then the minimum $\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$ is a Boolean uncertain variable such that*

$$\mathcal{M}\{\xi = 1\} = a_1 \wedge a_2 \wedge \cdots \wedge a_n.$$

The maximum $\eta = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\eta = 1\} = a_1 \vee a_2 \vee \cdots \vee a_n.$$

Theorem 5 *Assume that a system comprises of n components and each component is represented by an uncertain Boolean variable ξ_i such that $\mathcal{M}\{\xi_i = 1\} = R_i$, where $R_i, i = 1, 2, \ldots, n$ are the reliabilities of the components. Suppose the system has m minimum cut sets $C_1, C_2, \ldots, C_m$. If the failures of the components are independent from each other, the system's reliability will be*

$$R_S = \bigwedge_{i=1}^{m} \left\{ \bigvee_{j \in C_i} R_j \right\}. \tag{13}$$

Proof Let a Boolean uncertain variable ξ denote the state of the system (The system is working when $\xi = 1$). Since the system has m minimum cut sets $C_1, C_2, \ldots, C_m$,

$$\xi = \bigwedge_{i=1}^{m} \eta_i$$

where $\eta_i = \bigvee_{j \in C_i} \xi_j,\ i = 1, 2, \ldots, m$. From Theorem 4,

$$\mathcal{M}\{\eta_i = 1\} = \bigvee_{j \in C_i} \mathcal{M}\{\xi_j = 1\} = \bigvee_{j \in C_i} R_j.$$

Thus,

$$R_S = \mathcal{M}\{\xi = 1\} = \bigwedge_{i=1}^{m} \mathcal{M}\{\eta_i = 1\} = \bigwedge_{i=1}^{m} \left\{ \bigvee_{j \in C_i} R_j \right\}.$$

□

Algorithm 1 is a combination of MOCUS and Theorem 5, which tells how to conduct belief reliability evaluation for a time-static fault tree.
Algorithm 1 (Belief Reliability Evaluation for a Time-Static Fault Tree):

Step 1: Enumerate all minimum cut sets:
Loop from Top to Bottom:
if Gate == AND
  Increase the elements in each cut set;
if Gate == OR
  Increase the number of the cut sets.
Discard the cut sets which include other cut sets.
Step 2: Obtain system reliability from Theorem 5.
End.

Example 1 Consider a time-static system whose fault tree is given in Fig. 1. In the fault tree, top event T denotes the system's failure and x_1, x_2, x_3, x_4 denote the failures of the four components, respectively. Reliabilities of the components are

$$R_1 = 0.85,\ \ R_2 = 0.92,\ \ R_3 = 0.95,\ \ R_4 = 0.90.$$

Algorithm 1 is applied to evaluate the system's reliability.

Step 1: The minimum cut sets are

$$\{x_1\}, \{x_2, x_3\}, \{x_2, x_4\}.$$

Table 2 Life distributions of the components

Component	Life distribution
x_1	$\mathcal{L}(450, 550)$
x_2	$\mathcal{N}(500, 10)$
x_3	$\mathcal{Z}(450, 500, 550)$
x_4	$\mathcal{N}(500, 20)$

Step 2: From Theorem 5,

$$R_S = \min\{R_1, \max\{R_2, R_3\}, \max\{R_2, R_4\}\} = 0.85.$$

5.2 Time-dependent systems

In this section, results obtained in Sect. 5.1 will be used to evaluate a time-dependent system.

Theorem 6 (Evaluation of Time-Dependent Systems) *Let S be a time-dependent system whose time-static fault tree T has m minimum cut sets $C_1, C_2, \ldots, C_m$. Assume the system comprises of n components $X_1, X_2, \ldots, X_n$ and their belief reliable life functions are $BL_1(\alpha), BL_2(\alpha), \ldots, BL_n(\alpha)$, $0 < \alpha < 1$, respectively. Then the system's belief reliable life will be*

$$BL_S(\alpha) = \bigwedge_{i=1}^{m} \left\{ \bigvee_{j \in C_i} BL_j(\alpha) \right\}. \tag{14}$$

Proof Let T denote the system's life and t_i denote the life of component X_i. Since T can be obtained from the lives of elements in each minimum cut set by

$$T = \bigwedge_{i=1}^{m} \left\{ \bigvee_{j \in C_i} t_j \right\}$$

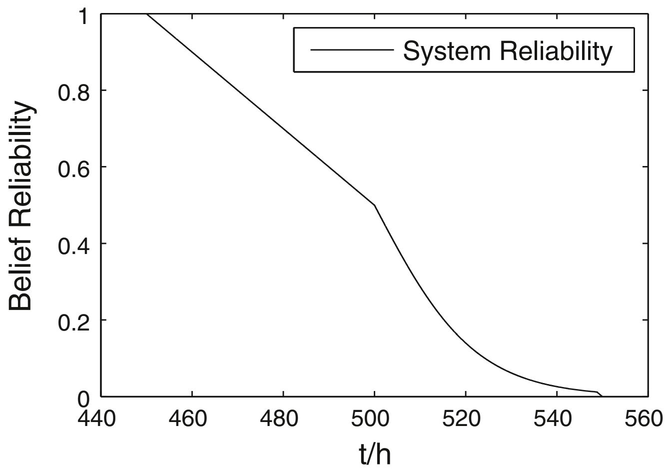
and $f(t_1, t_2, \ldots, t_n) = \bigwedge_{i=1}^{m} \left\{ \bigvee_{j \in C_i} t_j \right\}$ is a strictly increasing function with respect to $t_1, t_2, \ldots, t_n$, according to the operation law,

$$\Phi_{\mathrm{T}}^{-1}(\alpha) = \bigwedge_{i=1}^{m} \left\{ \bigvee_{j \in C_i} \Phi_{\mathrm{j}}^{-1}(\alpha) \right\}.$$

Thus Eq. (14) follows immediately from Theorem 1. □

Example 2 Consider a time-dependent system whose time-static fault tree is shown in Fig. 1. Life distributions of the components are listed in Table 2. Decide the belief reliability function of the system and the belief reliable life for $\alpha = 0.9$.

Fig. 2 Evaluation results



The three minimum cut sets are $\{x_1\}$, $\{x_2, x_3\}$, and $\{x_2, x_4\}$. So the system's belief reliable life $BL_S(\alpha)$ is obtained from Eq. (14) and the belief reliability function of the system is demonstrated in Fig. 2. For $\alpha = 0.9$, $BL(\alpha) = 460$ (h). Mean time to failure $MTTF_B$ and variance of life VL_B are achieved from Theorems 2 and 3, respectively.

$$MTTF_B = 494.97 \text{ (h)}, \; VL_B = 576.82 \text{ (h}^2\text{)}.$$

6 Conclusion

This paper focused on the application of uncertainty theory in reliability evaluation of systems with high reliability and long life.

1. Some new reliability metrics based on uncertainty theory were defined to characterize systems' reliability from different aspects.
2. Numerical methods for obtaining the defined metrics $MTTF_B$ and VL_B from belief reliable life $BL(\alpha)$ were proposed.
3. Fault tree analysis methods in the context of belief reliability were developed for numerical evaluation of complex systems.

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